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Value Distribution of Differential Polynomials Which Involve Two Distinct Transcendental Meromorphic Functions

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Abstract: *In the present note we deal with the value distribution of differential polynomials which involve two distinct transcendental meromorphic functions and obtain analogous results of Bhoosnurmath et al. [1] for such kind of differential polynomials.*

Keywords: Differential polynomials, Monomials, Transcendental meromorphic functions, Small functions, Value distribution

1 Introduction

We first consider the following definition used by S. S. Bhoosnurmath et al. in [1].

Definition 1. *Let $f(z)$ be a transcendental meromorphic function in the complex plane and let $m(r, f)$, $N(r, f)$, $T(r, f)$ have the usual meaning of Nevanlinna Theory. Let $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ except possibly a set of finite linear measure. Let $a_j(z)$ be meromorphic functions in the plane satisfying $T(r, a_j) = S(r, f)$ for $j = 1, 2, ..., n$; as $r \to \infty$. Then for a positive integer j, by a monomial in $f(z)$ we mean an expression of the form*

$$M_j[f] = a_j(z)[f(z)]^{n_{0j}}[f^{(1)}(z)]^{n_{1j}}...[f^{(k)}(z)]^{n_{kj}}$$

where n_{0j}, n_{1j}, ... , n_{kj} are non-negative integers. We define $d(M_j) = \sum_{i=0}^{k} n_{ij}$ as the degree of $M_j[f]$ and $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$ as the weight of $M_j[f]$.

Next, we define a differential polynomial in $f(z)$ as a finite sum of such monomials, i.e.,

$$P[f] = \sum_{j=1}^{n} a_j(z)M_j[f].$$

Also, we define

$$\bar{d}(P) = \max\{d(M_j) : 1 \le j \le n\},$$

$$\underline{d}(P) = \min\{d(M_j) : 1 \le j \le n\}$$

and

$$\Gamma_P = \max\{\Gamma_{M_j} : 1 \le j \le n\}$$

as the degree, the lower degree and the weight of $P[f]$ respectively.
If in particular, $\bar{d}(P) = \underline{d}(P)$, then $P[f]$ is called homogeneous and otherwise it is called non-homogeneous.

The study on value distribution of differential polynomials of meromorphic functions is a prominent field of research in Complex Analysis. A lot of research works [1, 2, 4, 5, 7, 8, 9, 10] have been done in this field by several mathematicians during the past decades.

In [1], Bhoosnurmath et al. proved the following theorems.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function with*

$$N(r,f)+N(r,\frac{1}{f})=S(r,f)$$

and let $P[f]$ be a differential polynomial in $f(z)$ of degree $\bar{d}(P)$ and lower degree $\underline{d}(P)$. Then

$$\underline{d}(P)\leq \underline{\lim}_{r\to+\infty}\frac{T(r,P[f])}{T(r,f)}\leq \overline{\lim}_{r\to+\infty}\frac{T(r,P[f])}{T(r,f)}\leq 2\bar{d}(P)-\underline{d}(P).$$

Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function with*

$$N(r,f)+N(r,\frac{1}{f})=S(r,f)$$

and let $P[f]$ be a differential polynomial in $f(z)$ of degree $\bar{d}(P)$ and lower degree $\underline{d}(P)$. Suppose that $P[f]$ does not reduce to a constant.

a. *If $P[f]$ is a homogeneous differential polynomial, then we have*

$$\delta(a,P[f])=0$$

for any $a\neq 0$, i.e., $P[f]$ assumes all finite complex values except possibly zero infinitely often.

b. *If $P[f]$ is a non-homogeneous differential polynomial with $2\underline{d}(P)>\bar{d}(P)$, then we have*

$$\delta(a,P[f])\leq 1-\frac{2\underline{d}(P)-\bar{d}(P)}{\underline{d}(P)}<1$$

for any $a\neq 0$, i.e., $P[f]$ assumes all finite complex values except possibly zero infinitely often.

The proofs of Theorem 1.1 and Theorem 1.2 are based on the following three lemmas.

Lemma 1.1. *[3] Let $f(z)$ be a meromorphic function and $P[f]$ be a differential polynomial with coefficients $a_j(z)$ and degree $\bar{d}(P)$ and lower degree $\underline{d}(P)$. Then*

$$m(r,\frac{P[f]}{f^{\bar{d}(P)}})\leq [\bar{d}(P)-\underline{d}(P)]m(r,\frac{1}{f})+S(r,f).$$

Lemma 1.2. *[6] Let $f(z)$ be meromorphic and non-constant in the plane. Then there are positive constants C_1 and C_2 such that*

$$m(r,\frac{f'}{f})\leq C_1\log r+C_2\log T(r,f)$$

as r tends to infinity outside possibly a set E of finite measure.

Lemma 1.3. *[1] Let $f(z)$ be a meromorphic function with a pole of order $p\geq 1$ at z_0. If $P[f]$ is a differential polynomial in $f(z)$ whose coefficients are analytic at z_0, then $P[f]$ has a pole at z_0 of order at most $p\bar{d}(P)+\Gamma_P-\bar{d}(P)$.*

Now we feel the need to find the answer of the simple and relevant query that what forms do the Theorem 1.1 and Theorem 1.2 take for differential polynomials of two transcendental meromorphic functions instead of just one. And to do that we first introduce the following analogous definition for homogeneous differential polynomials formed with two transcendental meromorphic functions f and g.

Definition 2. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and for positive integers $j=1,2,...,n$; let $a_j(z)$ be meromorphic functions in the plane satisfying $T(r,a_j)=S(r,f)$ and $T(r,a_j)=S(r,g)$ as $r\to\infty$. Then we define a monomial in $f(z)$ and $g(z)$ as an expression of the form*

$$M_j[f,g]=a_j(z)[f(z)]^{l_{0j}}[f^{(1)}(z)]^{l_{1j}}...[f^{(k)}(z)]^{l_{kj}}[g(z)]^{m_{0j}}[g^{(1)}(z)]^{m_{1j}}...[g^{(h)}(z)]^{m_{hj}}$$

where $l_{0j}, l_{1j}, \ldots, l_{kj}, m_{0j}, m_{1j}, \ldots, m_{hj}$ *are non-negative integers. We define*

$$d(M_j) = \sum_{i=0}^{k} l_{ij} + \sum_{i=0}^{h} m_{ij}$$

as the degree of $M_j[f,g]$ *and*

$$\Gamma_{M_j} = \sum_{i=0}^{k} (i+1) l_{ij} + \sum_{i=0}^{h} (i+1) m_{ij}$$

as the weight of $M_j[f,g]$.

Also, we define a differential polynomial in $f(z)$ and $g(z)$ as a finite sum of such monomials, i.e.,

$$P[f,g] = \sum_{j=1}^{n} a_j(z) M_j[f,g]. \tag{1}$$

Here also we define

$$\bar{d}(P) = \max\{d(M_j) : 1 \leq j \leq n\},$$
$$\underline{d}(P) = \min\{d(M_j) : 1 \leq j \leq n\}$$

and

$$\Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq n\}$$

as the degree, the lower degree and the weight of $P[f,g]$ respectively.
If in particular, $\bar{d}(P) = \underline{d}(P)$, then $P[f,g]$ is called homogeneous and otherwise it is called non-homogeneous.

In addition, for our convenience we define the followings:

$$\bar{d}_f(P) = \max\{\sum_{i=0}^{k} l_{ij} : 1 \leq j \leq n\};$$

$$\underline{d}_f(P) = \min\{\sum_{i=0}^{k} l_{ij} : 1 \leq j \leq n\};$$

$$\bar{d}_g(P) = \max\{\sum_{i=0}^{h} m_{ij} : 1 \leq j \leq n\}$$

and

$$\underline{d}_g(P) = \min\{\sum_{i=0}^{h} m_{ij} : 1 \leq j \leq n\}.$$

In particular, if there exists at least one $j(= j_1, say)$ such that

$$\bar{d}_f(P) = \sum_{i=0}^{k} l_{ij_1} \qquad and \qquad \bar{d}_g(P) = \sum_{i=0}^{h} m_{ij_1}, \tag{2}$$

then

$$\bar{d}(P) = \bar{d}_f(P) + \bar{d}_g(P).$$

Similarly if there exists at least one $j(= j_2, say)$ such that

$$\underline{d}_f(P) = \sum_{i=0}^{k} l_{ij_2} \qquad and \qquad \underline{d}_g(P) = \sum_{i=0}^{h} m_{ij_2}, \tag{3}$$

then

$$\underline{d}(P) = \underline{d}_f(P) + \underline{d}_g(P).$$

To clarify this particular scenario we can consider the following examples.

Example 1. *Let f and g be two transcendental meromorphic functions. We consider the differential polynomial*

$$P[f,g] = a_1 M_1[f,g] + a_2 M_2[f,g] + a_3 M_3[f,g],$$

where

$$M_1[f,g] = ff^{'}(f^{''})^2 g^2 (g^{''})^2, \quad M_2[f,g] = f^2 (f^{''})^3 g^2 (g^{'})^2, \quad M_3[f,g] = ff^{'}(f^{''})^2 g^{'}(g^{''})^2.$$

Here we have,

$$\bar{d}_f(P) = 5, \quad \bar{d}_g(P) = 4, \quad \underline{d}_f(P) = 4, \quad \underline{d}_g(P) = 3, \quad \bar{d}(P) = 9 \ \text{ and } \ \underline{d}(P) = 7.$$

Also, here we can see that $j_1 = 2$, which means both $\bar{d}_f(P)$ and $\bar{d}_g(P)$ are attained in $M_2[f,g]$ and $j_2 = 3$, which means both $\underline{d}_f(P)$ and $\underline{d}_g(P)$ are attained in $M_3[f,g]$. Hence $P[f,g]$ satisfies equations (2) and (3). Also we observe that

$$\bar{d}(P) = \bar{d}_f(P) + \bar{d}_g(P)$$

and

$$\underline{d}(P) = \underline{d}_f(P) + \underline{d}_g(P).$$

Example 2. *Let f and g be two transcendental meromorphic functions. We consider the differential polynomial*

$$P[f,g] = a_1 M_1[f,g] + a_2 M_2[f,g] + a_3 M_3[f,g],$$

where

$$M_1[f,g] = ff^{'}(f^{''})^2 g^2 g^{'}(g^{''})^2, \quad M_2[f,g] = f^2 f^{'}(f^{''})^3 g^2 (g^{'})^2, \quad M_3[f,g] = f^2 f^{'}(f^{''})^2 g^{'}(g^{''})^2.$$

Here we have,

$$\bar{d}_f(P) = 6, \quad \bar{d}_g(P) = 5, \quad \underline{d}_f(P) = 4, \quad \underline{d}_g(P) = 3, \quad \bar{d}(P) = 10 \ \text{ and } \ \underline{d}(P) = 8.$$

We see that $\bar{d}_f(P)$ is attained in $M_2[f,g]$ but $\bar{d}_g(P)$ is attained in $M_1[f,g]$, which means $\bar{d}_f(P)$ and $\bar{d}_g(P)$ are attained in different monomials. Also here $\underline{d}_f(P)$ is attained in $M_1[f,g]$ and $\underline{d}_g(P)$ is attained in $M_3[f,g]$. So, in this case $P[f,g]$ does not satisfy equation (2) and (3). Also

$$\bar{d}(P) \neq \bar{d}_f(P) + \bar{d}_g(P)$$

and

$$\underline{d}(P) \neq \underline{d}_f(P) + \underline{d}_g(P).$$

Example 3. *Let f and g be two transcendental meromorphic functions. We consider the differential polynomial*

$$P[f,g] = a_1 M_1[f,g] + a_2 M_2[f,g] + a_3 M_3[f,g],$$

where

$$M_1[f,g] = f^2 f^{'}(f^{''})^2 g^2 (g^{''})^2, \quad M_2[f,g] = f^2 (f^{''})^3 g^2 (g^{'})^2, \quad M_3[f,g] = f(f^{'})^2 (f^{''})^2 g g^{'}(g^{''})^2.$$

Here we have,

$$\bar{d}_f(P) = \underline{d}_f(P) = 5 \ \text{ and } \ \bar{d}_g(P) = \underline{d}_g(P) = 4.$$

In fact, all of $\bar{d}_f(P)$, $\bar{d}_g(P)$, $\underline{d}_f(P)$ and $\underline{d}_g(P)$ are attained in each of the three monomials. Hence $P[f,g]$ satisfies equation (2) and (3). Here also we can observe that

$$\bar{d}(P) = 9 = \bar{d}_f(P) + \bar{d}_g(P)$$

and

$$\underline{d}(P) = 9 = \underline{d}_f(P) + \underline{d}_g(P),$$

i.e.,

$$\bar{d}(P) = \underline{d}(P),$$

which means $P[f,g]$ is homogeneous in this case.

As mentioned earlier, our main goal is to extend the results of Theorem 1.1 and Theorem 1.2 for differential polynomials formed with two transcendental meromorphic functions instead of one meromorphic function by imposing certain conditions. Further we see that some of these conditions are sufficient but not necessary.

Here we do not explain the standard notations and definitions of the value distribution theory as those are available in [6].

2 Lemmas

In this section, we prove two lemmas which will be needed in the sequel.

Lemma 2.1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and $P[f,g]$ be a differential polynomial defined by equation (1) with degree $\bar{d}(P)$ and lower degree $\underline{d}(P)$. If $P[f,g]$ satisfies equations (2) and (3) then*

$$m(r, \frac{P[f,g]}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}}) \leq n[\bar{d}(P) - \underline{d}(P)] \times [m(r, \frac{1}{f}) + m(r, \frac{1}{g})] + S(r,f) + S(r,g).$$

Proof. By Lemma 1.2 and the fact that $T(r,a_j) = S(r,f)$ and $T(r,a_j) = S(r,g)$ for $j = 1,2,...,n$, we have

$$\begin{aligned}
m(r, \frac{P[f,g]}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}}) &= m(r, \frac{\sum_{j=1}^n a_j(z)M_j[f,g]}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}}) \\
&= m(r, \frac{\sum_{j=1}^n a_j(z)[f(z)]^{l_{0j}}[f^{(1)}(z)]^{l_{1j}}...[f^{(k)}(z)]^{l_{kj}}[g(z)]^{m_{0j}}[g^{(1)}(z)]^{m_{1j}}...[g^{(h)}(z)]^{m_{hj}}}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}}) \\
&= m(r, \sum_{j=1}^n a_j(z)[\frac{f^{(1)}(z)}{f(z)}]^{l_{1j}}...[\frac{f^{(k)}(z)}{f(z)}]^{l_{kj}}[\frac{g^{(1)}(z)}{g(z)}]^{m_{1j}}...[\frac{g^{(h)}(z)}{g(z)}]^{m_{hj}} \\
&\qquad \times \frac{1}{(f)^{\bar{d}_f(P)-\sum_{i=0}^k l_{ij}} \times (g)^{\bar{d}_g(P)-\sum_{i=0}^h m_{ij}}})
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^n m(r, a_j(z)[\frac{f^{(1)}(z)}{f(z)}]^{l_{1j}}...[\frac{f^{(k)}(z)}{f(z)}]^{l_{kj}}[\frac{g^{(1)}(z)}{g(z)}]^{m_{1j}}...[\frac{g^{(h)}(z)}{g(z)}]^{m_{hj}} \\
&\qquad \times \frac{1}{(f)^{\bar{d}_f(P)-\sum_{i=0}^k l_{ij}} \times (g)^{\bar{d}_g(P)-\sum_{i=0}^h m_{ij}}}) \\
&\leq \sum_{j=1}^n [m(r, a_j(z)) + l_{1j}m(r, \frac{f^{(1)}(z)}{f(z)}) + ... + l_{kj}m(r, \frac{f^{(k)}(z)}{f(z)}) + m_{1j}m(\frac{g^{(1)}(z)}{g(z)}) + ... \\
&\qquad ... + m_{hj}m(r, \frac{g^{(h)}(z)}{g(z)}) + m(r, \frac{1}{(f)^{\bar{d}_f(P)-\sum_{i=0}^k l_{ij}}}) + m(r, \frac{1}{(g)^{\bar{d}_g(P)-\sum_{i=0}^h m_{ij}}})] + O(1) \\
&\leq \sum_{j=1}^n [(\bar{d}_f(P) - \sum_{i=0}^k l_{ij})m(r, \frac{1}{f}) + (\bar{d}_g(P) - \sum_{i=0}^h m_{ij})m(r, \frac{1}{g})] + S(r,f) + S(r,g) \\
&\leq \sum_{j=1}^n [(\bar{d}_f(P) - \underline{d}_f(P))m(r, \frac{1}{f}) + (\bar{d}_g(P) - \underline{d}_g(P))m(r, \frac{1}{g})] + S(r,f) + S(r,g) \\
&\leq n[\bar{d}_f(P) - \underline{d}_f(P)]m(r, \frac{1}{f}) + n[\bar{d}_g(P) - \underline{d}_g(P)]m(r, \frac{1}{g}) + S(r,f) + S(r,g) \\
&\leq n[\bar{d}_f(P) - \underline{d}_f(P)][m(r, \frac{1}{f}) + m(r, \frac{1}{g})] + n[\bar{d}_g(P) - \underline{d}_g(P)][m(r, \frac{1}{f}) + m(r, \frac{1}{g})] \\
&\qquad + S(r,f) + S(r,g) \\
&\leq n[\{\bar{d}_f(P) + \bar{d}_g(P)\} - \{\underline{d}_f(P) + \underline{d}_g(P)\}][m(r, \frac{1}{f}) + m(r, \frac{1}{g})] + S(r,f) + S(r,g) \\
&= n[\bar{d}(P) - \underline{d}(P)] \times [m(r, \frac{1}{f}) + m(r, \frac{1}{g})] + S(r,f) + S(r,g).
\end{aligned}$$

□

Lemma 2.2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with a pole of order p and q respectively at z_0, where at least one of p and q is non-zero. If $P[f,g]$ is a differential polynomial in $f(z)$ and $g(z)$ whose coefficients are analytic at z_0, then $P[f,g]$ has a pole at z_0 of order at most $(p+q)\bar{d}(P)+\Gamma_P-\bar{d}(P)$*

Proof. $P[f,g]$ being a differential polynomial in $f(z)$ and $g(z)$ is a finite sum of terms of the form

$$a_j(z)[f(z)]^{l_{0j}}[f^{(1)}(z)]^{l_{1j}}...[f^{(k)}(z)]^{l_{kj}}[g(z)]^{m_{0j}}[g^{(1)}(z)]^{m_{1j}}...[g^{(h)}(z)]^{m_{hj}},$$

where $a_j(z)$ is analytic at z_0. Each of this term will have a pole at z_0 of order at most

$$\begin{aligned}
&\max_{1\le j\le n}[\{pl_{0j}+(p+1)l_{1j}+...+(p+k)l_{kj}\}+\{qm_{0j}+(q+1)m_{1j}+...+(q+h)m_{hj}\}]\\
&=\max_{1\le j\le n}\{p(l_{0j}+l_{1j}+...+l_{kj})+(l_{1j}+2l_{2j}+...+kl_{kj})+q(m_{0j}+m_{1j}+...+m_{hj})\\
&\qquad\qquad+(m_{1j}+2m_{2j}+...+hm_{hj})\}\\
&=\max_{1\le j\le n}[(p-1)(l_{0j}+l_{1j}+...+l_{kj})+\{l_{0j}+2l_{1j}+...+(k+1)l_{kj}\}\\
&\qquad\qquad+(q-1)(m_{0j}+m_{1j}+...+m_{hj})+\{m_{0j}+2m_{1j}+...+(h+1)m_{hj}\}]\\
&=\max_{1\le j\le n}[(p-1)\sum_{i=0}^{k}l_{ij}+\sum_{i=0}^{k}(i+1)l_{ij}+(q-1)\sum_{i=0}^{h}m_{ij}+\sum_{i=0}^{h}(i+1)m_{ij}]\\
&\le(p+q-1)\max_{1\le j\le n}\{\sum_{i=0}^{k}l_{ij}+\sum_{i=0}^{h}m_{ij}\}+\max_{1\le j\le n}\{\sum_{i=0}^{k}(i+1)l_{ij}+\sum_{i=0}^{h}(i+1)m_{ij}\}\\
&=(p+q-1)\bar{d}(P)+\Gamma_P\\
&=(p+q)\bar{d}(P)+\Gamma_P-\bar{d}(P).
\end{aligned}$$

Hence $P[f,g]$ has a pole at z_0 of order at most $(p+q)\bar{d}(P)+\Gamma_P-\bar{d}(P)$. □

3 Main Results

In this section we present our main results of the paper.

Theorem 3.1. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions satisfying*

$$N(r,f)+N(r,\frac{1}{f})=S(r,f) \tag{4}$$

and

$$N(r,g)+N(r,\frac{1}{g})=S(r,g) \tag{5}$$

and let $P[f,g]$ be a differential polynomial defined by equation (1) with degree $\bar{d}(P)$ and lower degree $\underline{d}(P)$. Also let $P[f,g]$ satisfies equations (2) and (3). Then

$$\bar{\lim}_{r\to+\infty}\frac{T(r,P[f,g])}{T(r,f)+T(r,g)}\le(n+1)\bar{d}(P)-n\underline{d}(P).$$

Proof. The poles of $P[f,g]$ can occur only at the poles of f and g or at the poles of the coefficients $a_j(z)$ of $P[f,g]$. Now since a_j's are small functions of f and g, we can ignore the poles of the coefficients $a_j(z)$. Hence, if z_0 be a pole of f and g of order p and q respectively, then by Lemma 2.2 we can say that $P[f,g]$ will have a pole at z_0 of order at most

$$(p+q)\bar{d}(P)+\Gamma_P-\bar{d}(P).$$

Thus we have,

$$N(r,P[f,g])\le\bar{d}(P)\{N(r,f)+N(r,g)\}+\{\Gamma_P-\bar{d}(P)\}\{\bar{N}(r,f)+\bar{N}(r,g)\}+S(r,f)+S(r,g). \tag{6}$$

Then by using (4), (5) and (6), we have

$$\begin{aligned}
N(r, \frac{P[f,g]}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}}) &\le N(r, P[f,g]) + N(r, \frac{1}{f^{\bar{d}_f(P)}}) + N(r, \frac{1}{g^{\bar{d}_g(P)}}) \\
&\le \bar{d}(P)\{N(r,f) + N(r,g)\} + \{\Gamma_P - \bar{d}(P)\}\{\bar{N}(r,f) + \bar{N}(r,g)\} \\
&\qquad + \bar{d}_f(P)N(r, \frac{1}{f}) + \bar{d}_g(P)N(r, \frac{1}{g}) + S(r,f) + S(r,g) \\
&\le \bar{d}(P)\{N(r,f) + N(r,g)\} + \{\Gamma_P - \bar{d}(P)\}\{\bar{N}(r,f) + \bar{N}(r,g)\} \\
&\qquad + \bar{d}(P)N(r, \frac{1}{f}) + \bar{d}(P)N(r, \frac{1}{g}) + S(r,f) + S(r,g) \\
&\le \bar{d}(P)\{N(r,f) + N(r, \frac{1}{f})\} + \bar{d}(P)\{N(r,g) + N(r, \frac{1}{g})\} \\
&\qquad + \{\Gamma_P - \bar{d}(P)\}\{\bar{N}(r,f) + \bar{N}(r,g)\} + S(r,f) + S(r,g) \\
&= S(r,f) + S(r,g). \qquad (7)
\end{aligned}$$

Thus by using First Fundamental Theorem, Lemma 2.1 and (7), we have

$$\begin{aligned}
T(r, P[f,g]) &\le T(r, \frac{P[f,g]}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}}) + T(r, f^{\bar{d}_f(P)}) + T(r, g^{\bar{d}_g(P)}) \\
&\le m(r, \frac{P[f,g]}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}}) + N(r, \frac{P[f,g]}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}}) + \bar{d}_f(P)T(r,f) + \bar{d}_g(P)T(r,g) \\
&\le n[\bar{d}(P) - \underline{d}(P)] \times [m(r, \frac{1}{f}) + m(r, \frac{1}{g})] + \bar{d}(P)T(r,f) + \bar{d}(P)T(r,g) \\
&\qquad + S(r,f) + S(r,g) \\
&\le n[\bar{d}(P) - \underline{d}(P)] \times [T(r, \frac{1}{f}) + T(r, \frac{1}{g})] + \bar{d}(P)T(r,f) + \bar{d}(P)T(r,g) \\
&\qquad + S(r,f) + S(r,g) \\
&= [(n+1)\bar{d}(P) - n\underline{d}(P)] \times [T(r,f) + T(r,g)] + S(r,f) + S(r,g). \qquad (8)
\end{aligned}$$

And finally inequality (8) implies that

$$\overline{\lim}_{r\to+\infty} \frac{T(r, P[f,g])}{T(r,f) + T(r,g)} \le (n+1)\bar{d}(P) - n\underline{d}(P). \qquad (9)$$

□

Theorem 3.2. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions satisfying equations (4) and (5) respectively and let $P[f,g]$ be a homogeneous differential polynomial. If $P[f,g]$ satisfies equations (2) and (3) and does not reduce to a constant, then*

$$\delta(a, P[f,g]) = 0$$

for any $a \neq 0$.

Proof. We have from equation (8)

$$T(r, P[f,g]) \le K[T(r,f) + T(r,g)] + o(T(r,f)) + o(T(r,g)) = [K + o(1)][T(r,f) + T(r,g)]).$$

Therefore

$$\begin{aligned}
S(r, P[f,g]) &= o(T(r, P[f,g])) \\
&= o(1)[K + o(1)][T(r,f) + T(r,g)] \\
&= o(T(r,f)) + o(T(r,g)) \\
&= S(r,f) + S(r,g). \qquad (10)
\end{aligned}$$

Again by using equations (4), (5), (6) and (10), we have

$$\begin{aligned}N(r,P[f,g]) &\leq \bar{d}(P)\{N(r,f)+N(r,g)\}+\{\Gamma_P-\bar{d}(P)\}\{N(r,f)+N(r,g)\}+S(r,f)+S(r,g)\\ &\leq \Gamma_P\{N(r,f)+N(r,g)\}+S(r,f)+S(r,g)\\ &\leq \Gamma_P\{S(r,f)+S(r,g)\}+S(r,f)+S(r,g)\\ &= S(r,f)+S(r,g)\\ &= S(r,P[f,g]). \qquad (11)\end{aligned}$$

Also using First Fundamental Theorem, we have

$$\begin{aligned}\bar{N}(r,\frac{1}{P[f,g]}) &\leq \bar{N}(r,\frac{1}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}})+\bar{N}(r,\frac{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}}{P[f,g]})\\ &\leq \bar{N}(r,\frac{1}{fg})+T(r,\frac{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}}{P[f,g]})\\ &\leq \bar{N}(r,\frac{1}{fg})+T(r,\frac{P[f,g]}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}})+O(1). \qquad (12)\end{aligned}$$

Now by using Lemma2.1, (7), (10) and since $P[f,g]$ is a homogeneous differential polynomial, we have

$$\begin{aligned}T(r,\frac{P[f,g]}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}}) &= m(r,\frac{P[f,g]}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}})+N(r,\frac{P[f,g]}{f^{\bar{d}_f(P)}g^{\bar{d}_g(P)}})\\ &\leq n[\bar{d}(P)-\underline{d}(P)]\times[m(r,\frac{1}{f})+m(r,\frac{1}{g})]+S(r,f)+S(r,g)\\ &= S(r,P[f,g]). \qquad (13)\end{aligned}$$

Hence using (4), (5), (10) and (13), we can rewrite (12) as

$$\begin{aligned}\bar{N}(r,\frac{1}{P[f,g]}) &\leq \bar{N}(r,\frac{1}{fg})+S(r,P[f,g])\\ &\leq N(r,\frac{1}{f})+N(r,\frac{1}{g})+S(r,P[f,g])\\ &\leq S(r,P[f,g]). \qquad (14)\end{aligned}$$

Now if $a \neq 0$, then by Second Fundamental Theorem, (11) and (14), we have

$$\begin{aligned}T(r,P[f,g]) &\leq \bar{N}(r,P[f,g])+\bar{N}(r,\frac{1}{P[f,g]})+\bar{N}(r,\frac{1}{P[f,g]-a})+S(r,P[f,g])\\ &\leq \bar{N}(r,\frac{1}{P[f,g]-a})+S(r,P[f,g]),\end{aligned}$$

i.e.,

$$[1+o(1)]T(r,P[f,g]) \leq \bar{N}(r,\frac{1}{P[f,g]-a}).$$

Thus we have for large r outside possibly a set E of finite linear measure

$$[1-\frac{\bar{N}(r,\frac{1}{P[f,g]-a})}{T(r,P[f,g])}]+o(1) \leq 0$$

which gives

$$\delta(a,P[f,g])=0.$$

□

Example 4. *Let $f(z) = e^z$ and $g(z) = e^{-2z}$ and consider the following homogeneous differential polynomial in f and g.*

$$P[f, g] = f(f')^2 g + (f')^3 g'$$

i.e.,

$$\begin{aligned} P[f, g] &= e^z.(e^z)^2.e^{-2z} + (e^z)^3.(-2)e^{-2z} \\ &= e^z - 2e^z \\ &= -e^z. \end{aligned}$$

Now for $a = 0$, we have

$$\begin{aligned} N(r, \frac{1}{P[f, g] - a}) &= N(r, \frac{1}{-e^z - 0}) \\ &= 0 \quad as \quad r \to \infty. \end{aligned}$$

Hence

$$\begin{aligned} \delta(a, P[f, g]) &= \overline{\lim}_{r\to\infty}[1 - \frac{N(r, \frac{1}{P[f,g]-a})}{T(r, P[f, g])}] \\ &= 1. \end{aligned}$$

So, Theorem 3.2 does not hold here as we have assumed $a = 0$. So, the condition that $a \neq 0$ in Theorem 3.2 is not redundant.

Now we can see the following examples which suggests that the conditions mentioned in Theorem 3.2 are sufficient but not necessary.

Example 5. *Let $f(z) = 3z + 2$ and $g(z) = z^2 + 3z - 4$ and consider the following homogeneous differential polynomial in f and g.*

$$P[f, g] = ff'g + (f')^2 g'$$

i.e.,

$$\begin{aligned} P[f, g] &= 3(3z + 2)(z^2 + 3z - 4) + 9(2z + 3) \\ &= 9z^3 + 33z^2 + 3. \end{aligned}$$

Now for $a = 3$, we have

$$\begin{aligned} m(r, \frac{1}{P[f, g] - a}) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|9r^3e^{3i\theta} + 33r^2e^{2i\theta}|} d\theta \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{9r^3 - 33r^2} d\theta \to 0 \quad as \quad r \to \infty. \end{aligned}$$

Hence

$$\begin{aligned} \delta(a, P[f, g]) &= \overline{\lim}_{r\to\infty}[1 - \frac{N(r, \frac{1}{P[f,g]-a})}{T(r, P[f, g])}] \\ &= 0. \end{aligned}$$

So, Theorem 3.2 holds although f and g are not transcendental.

Example 6. *Let $f(z) = ze^z$ and $g(z) = e^{-2z}$ and consider the following non-homogeneous differential polynomial in f and g.*

$$P[f, g] = ff'g + (f)^4(g')^2$$

i.e.,

$$\begin{aligned} P[f,g] &= ze^z.(ze^z+e^z).e^{-2z}+(ze^z)^4(-2e^{-2z})^2 \\ &= z(z+1)+4z^4 \\ &= 4z^4+z^2+z. \end{aligned}$$

Now for $a=0$, we have

$$\begin{aligned} m(r,\frac{1}{P[f,g]-a}) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|4r^4e^{4i\theta}+r^2e^{2i\theta}+re^{i\theta}|}d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{4r^4-r^2-r}d\theta \to 0 \quad as \quad r\to\infty. \end{aligned}$$

Hence

$$\begin{aligned} \delta(a,P[f,g]) &= \overline{\lim}_{r\to\infty}[1-\frac{N(r,\frac{1}{P[f,g]-a})}{T(r,P[f,g])}] \\ &= 0. \end{aligned}$$

So, Theorem 3.2 holds although $P[f,g]$ is non-homogeneous and $a=0$.

Example 7. *Let $f(z)=ze^z$ and $g(z)=e^{-2z}$ and consider the following homogeneous differential polynomial in f and g.*

$$P[f,g]=ff'g+fgg'+2fg^2$$

i.e.,

$$\begin{aligned} P[f,g] &= ze^z.(ze^z+e^z).e^{-2z}+ze^z.e^{-2z}.(-2e^{-2z})+2.ze^z.(e^{-2z})^2 \\ &= z(z+1)-2ze^{-3z}+2ze^{-3z} \\ &= z^2+z. \end{aligned}$$

Here it is clear that $P[f,g]$ neither satisfies (2) nor (3). Yet for any a, we have

$$\begin{aligned} m(r,\frac{1}{P[f,g]-a}) &= \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|r^2e^{2i\theta}+re^{i\theta}-a|}d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{r^2-r-|a|}d\theta \to 0 \quad as \quad r\to\infty. \end{aligned}$$

Hence

$$\begin{aligned} \delta(a,P[f,g]) &= \overline{\lim}_{r\to\infty}[1-\frac{N(r,\frac{1}{P[f,g]-a})}{T(r,P[f,g])}] \\ &= 0. \end{aligned}$$

So, Theorem 3.2 holds although $P[f,g]$ does not satisfy (2) and (3).

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