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# On Codes Over Mixed Finite Ring $\mathbb{Z}_2\mathbb{Z}_4$ and Its Related Parameter

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**Abstract:** *In this paper, we have to find out the lower bound and upper bound of some class codes in the finite ring $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$, where $u^3 = 0$, with Gray Mapping $\mathbb{R} \to \mathbb{Z}_2\mathbb{Z}_4$ with respect to different weight by using covering radius. Also, the covering radius of various lengths and different lengths of the Block Repetition Codes in the finite ring $\mathbb{R}$ is determined.*

**Keywords:** Finite ring, Covering radius, Repetition codes, Lee, Generalized Lee, Chinese Euclidean weights

## 1 Introduction

In the last decade, there have been many researchers doing research on code over finite rings. In [3], the author was the first to develop the coding theory of finite commutative non-chain rings. $\mathbb{Z}_{2k}$ is a special type of ring and $2k$ is the ring of integers modulo $2k$, $k$ is a positive integer, worked very much interest in codes over finite rings in recent years.

In[2, 4, 5, 6, 7], the authors widely study the codes over $\mathbb{Z}_4$ and get a good binary linear code and non-linear code over the finite ring $\mathbb{Z}_4$ via the gray map.

In [13], the author studied the covering radius of binary linear codes over finite fields and the covering radius is one of the important geometric parameters of codes. Recently, the covering radius of codes over finite chain rings has been studied. In 1999, Sole et al gave many upper and lower bounds on the covering radius of a code over $\mathbb{Z}_4$ with different distances.

In [10], the author studied the covering radius of codes over $\mathbb{Z}_2 + u\mathbb{Z}_2$ with $u^2 = 0$. The author gave some lower bound and upper bound on the covering radius of codes over $\mathbb{Z}_4$ and $\mathbb{Z}_2 + u\mathbb{Z}_2$ [8, 9, 11]. The Generalized Lee weight and Lee weight are the element $x \in \mathbb{R}$ is analogous to the definition of the Generalized Lee weight and Lee weight of the elements of the ring $\mathbb{Z}_8$ [12, 16, 17].

In continuation of the above work. I have given some results on the covering radius of codes in $\mathbb{R} = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ being the finite ring. Consider the elements of ring $\{0, 1, u, v, u^2, v^2, uv, v^3\}$, where $u^3 = 0, v = 1 + u, u^2 = 2u, v^2 = 1 + u^2, uv = u + u^2, v^3 = 1 + u + u^2$ and $\mathbb{Z}_2\mathbb{Z}_4 = \{00, 01, 02, 03, 10, 11, 12, 13\}$. That is, Gray Map $\mathbb{R} \to \mathbb{Z}_2\mathbb{Z}_4$.

## 2 Preliminaries

Let $C$ be a linear code of length $n$ in $\mathbb{R}$ is an additive subgroup of $\mathbb{R}^n$. An element of $C$ is called a *codeword* of $C$. A matrix whose rows are the basis elements of a linear code $C$ is said to be generator matrix of $C$. The *Hamming weight* of $C$ is $wt_H(C) = \{wt(c) | c \in C \text{ and } c \neq 0\}$. Let $c_1, c_2 \in C$ and $c_1 - c_2 \in C$. The Hamming distance of $C$ is $d_H(C) = \{d(c_1, c_2) | c_1, c_2 \in C \text{ and } c_1 \neq c_2\} = \{wt(c_1 - c_2) | c_1, c_2 \in C \text{ and } c_1 \neq c_2\} = \{wt(c) | c \in C \text{ and } c \neq 0\} = wt_H(C)$.

Any code $\mathbb{R}$ is permutation equivalent to a code $C$ with generator matrix of the form

$$G = \begin{bmatrix} I_{k_0} & A_{01} & A_{02} & A_{03} \\ 0 & uI_{k_1} & uA_{12} & uA_{13} \\ 0 & 0 & u^2 I_{k_2} & u^2 A_{23} \end{bmatrix}$$

where $A_{ij}$ are binary matrices for $i > 0$. A code with a generator matrix in this form is of type $\{k_0, k_1, k_2\}$ and has $8^{k_0}4^{k_1}2^{k_2}$ vectors [18].

The *Lee weights* of the elements $0, \{1, v, v^2, v^3\}, \{u, uv\}, u^2$ of $\mathbb{R}$ are defined by $0, 1, 2, 2^2$ [17]. In[1], the *Generalized Lee weight* of the elements $x \in \mathbb{R}$ are given

$$wt_{GL}(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 2 & \text{if} \quad x \neq u^2 \\ 4 & \text{if} \quad x = u^2 \end{cases}$$

and the Chinese Euclidean Weight of the elements $x \in \mathbb{R}$ are given

$$wt_{CE}(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ 1 & \text{if} \quad x = 1, v^3, \\ 2 & \text{if} \quad x = u, uv, \\ 3 & \text{if} \quad x = v, v^2, \\ 4 & \text{if} \quad x = u^2. \end{cases}$$

in [15].
The Lee, Generalized Lee and Chinese Euclidean distances between the codewords $c_1$ and $c_2 \in \mathbb{R}^n$ are defined as

$$d_L(c_1, c_2) = wt_L(c_1 - c_2),$$
$$d_{GL}(c_1, c_2) = wt_{GL}(c_1 - c_2)$$

and

$$d_{CE}(c_1, c_2) = wt_{CE}(c_1 - c_2).$$

The minimum Hamming weight of $C$ is $wt_H(C) = \min\{wt(c)|c \in C \text{ and } c \neq 0\}$. Similarly, the minimum Lee, minimum Generalized Lee and minimum Chinese Euclidean weights of $C$ is the smallest non-zero codeword of a code $C$.

The Gray map $\phi : \mathbb{R} \to \mathbb{Z}_2\mathbb{Z}_4$ is defined by $\phi(0) = (0, 0), \phi(1) = (1, 1), \phi(u) = (0, 2), \phi(v) = (0, 3), \phi(u^2) = (1, 0), \phi(v^2) = (0, 1), \phi(uv) = (1, 2)$ and $\phi(v^3) = (1, 3)$. In general, a linear gray map $\phi$ from $\mathbb{R}^n \to \mathbb{Z}_2^n \times \mathbb{Z}_4^n$ is the coordinates-wise extension of the function from $\mathbb{R}$ to $\mathbb{Z}_2\mathbb{Z}_4$.

**Example 1.** *Let $c = 0\ 1\ u\ v \in \mathbb{R}$ be a codeword of code. Find Lee weight, Generalized Lee weight and Chinese Euclidean weight of $c$*

1. $d_L(c) = d_L(0\ 1\ u\ v) = d_L(0) + d_L(1) + d_L(u) + d_L(v) = 0 + 1 + 2 + 2 = 5.$
2. $d_{GL}(c) = d_{GL}(0\ 1\ u\ v) = d_{GL}(0) + d_{GL}(1) + d_{GL}(u) + d_{GL}(v) = 0 + 2 + 2 + 2 = 6.$
3. $d_{CE}(c) = d_{CE}(0\ 1\ u\ v) = d_{Ce}(0) + d_{CE}(1) + d_{CE}(u) + d_{CE}(v) = 0 + 1 + 2 + 3 = 6.$

# 3 Covering Radius and Repetition Codes $\mathbb{R}$

Let $d$ be the distance of the codeword of a code $C$ in $\mathbb{R}$ with respect to Hamming weight, Lee weight, Generalized Lee weight and Chinese Euclidean weight. The *covering radius* of a code $C$ in $\mathbb{R}$ is given by

$$r_d(C) = \max_{r \in \mathbb{R}^n} \left\{ \min_{c \in C} \{d(r, c)\} \right\}.$$

Computing covering radius of codes in $\mathbb{R}$, for useful, the Mattson result in [13].

Let $C$ be the $q$-ary repetition code over a finite field $\mathbb{F}_q = \{\alpha_0 = 0, \alpha_1 = 1, \alpha_2, \alpha_3, \cdots, \alpha_{q-1}\}$ and $C = \{\bar{\alpha}|\alpha \in \mathbb{F}_q\}$, where $\bar{\alpha} = \alpha\alpha \cdots \alpha$ and it's the parameter of $C$ is an $[n, 1, n]$ code. In [14], the covering radius of $C$ is $\lfloor \frac{n(q-1)}{q} \rfloor$. Using above result, it can be found that the covering radius of block of size $n$ repetition code $[n(q-1), 1, n(q-1)]$ generated by $G = [\overbrace{11 \cdots 1}^{n} \overbrace{\alpha_2\alpha_2 \cdots \alpha_2}^{n} \cdots \overbrace{\alpha_{q-1}\alpha_{q-1} \cdots \alpha_{q-1}}^{n}]$ is $\lfloor \frac{n(q-1)^2}{q} \rfloor$, since it will be equivalent to a repetition code of length $(q-1)n$.
In $\mathbb{R}$, there are two types of repetition codes of length $n$ viz.

1. unit repetition code $C_I : [n, 1, d_H = n, d_L = n, d_{GL} = n, d_{CE} = n]$ generated by $G_I = [\overbrace{11\cdots 1}^{n}]$
2. zero repetition code $C_{II} : (n, 2, d_H = n, d_L = 4n, d_{GL} = 4n, d_{CE} = 4n)$ generated by $G_{II} = [\overbrace{u^2u^2\cdots u^2}^{n}]$ and $C_{III} : (n, 4, d_H = n, d_L = 2n, d_{GL} = 2n, d_{CE} = 2n)$ generated by $G_{III} = [\overbrace{u\ uv\ u\ uv\cdots u\ uv}^{n}]$ or $[\overbrace{uv\ u\ uv\ u\cdots uv\ u}^{n}]$. The code generated by $[u\ u\cdots u]$ and $[uv\ uv\cdots uv]$ are equivalent to the code $C_{III}$.

**Theorem 3.1.** *1.* $r_L(C_I) = \frac{3n}{2}$,

*2.* $r_L(C_{II}) = 2n$ *and*

*3.* $n \le r_L(C_{III}) \le 2n$.

*Proof.* If $x \in \mathbb{R}^n$ with $\omega_0$ times $0'$s, $\omega_1$ times $1'$s, $\omega_2$ times $2'$s, $\omega_3$ times $3'$s $\omega_4$ times $4'$s, $\omega_5$ times $5'$s, $\omega_6$ times $6'$s and $\omega_7$ times $7'$s in $x$ and $\omega_0+\omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6+\omega_7 = n$. The code $c_i \in \{\alpha(C_I)|\alpha \in \mathbb{R}\}, i = 0$ to 7. Then

$$\begin{aligned} d_L(x, c_0) &= wt_L(x - 00\cdots 0) \\ &= 0\omega_0 + 1\omega_1 + u\omega_2 + v\omega_3 + u^2\omega_4 + uv\omega_5 + v^2\omega_6 + v^3\omega_7 \\ d_L(x, c_0) &= n - \omega_0 + \omega_2 + 3\omega_4 + \omega_6. \\ d_L(x, c_1) &= wt_L(x - 11\cdots 1) \\ &= v^3\omega_0 + 0\omega_1 + 1\omega_2 + u\omega_3 + v\omega_4 + u^2\omega_5 + uv\omega_6 + v^2\omega_7 \\ d_L(x, c_1) &= n - \omega_1 + \omega_3 + 3\omega_5 + \omega_7. \end{aligned}$$

Similarly,

$$d_L(x, c_2) = n - \omega_2 + \omega_0 + \omega_4 + 3\omega_6,$$

$$d_L(x, c_3) = n - \omega_3 + \omega_5 + 3\omega_7 + \omega_1,$$

$$d_L(x, c_4) = n - \omega_4 + 3\omega_0 + \omega_2 + \omega_6$$

$$d_L(x, c_5) = n - \omega_5 + \omega_7 + 3\omega_1 + \omega_3,$$

$$d_L(x, c_6) = n - \omega_6 + \omega_0 + 3\omega_2 + \omega_4$$

and

$$d_L(x, c_7) = n - \omega_7 + \omega_1 + 3\omega_3 + \omega_5.$$

Therefore, $d_L(x, C_I) = \min\{d_L(x, c_0), d_L(x, c_1), d_L(x, c_2), d_L(x, c_3), d_L(x, c_4), d_L(x, c_5), d_L(x, c_6), d_L(x, c_7)\}$.

Since the minimum of data is less than or equal to the average of data and $\omega_0 + \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7 = n$, implies $d_L(x, C_I) \le n + \frac{4n}{8} = \frac{3n}{2}$. Thus, $r_L(C_I) \le \frac{3n}{2}$.

Let $x = \overbrace{00\cdots 0}^{t}\overbrace{11\cdots 1}^{t}\overbrace{uu\cdots u}^{t}\overbrace{vv\cdots v}^{t}\overbrace{u^2u^2\cdots u^2}^{t}\overbrace{v^2v^2\cdots v^2}^{t}$ $\overbrace{uvuv\cdots uv}^{t}\overbrace{v^3v^3\cdots v^3}^{n-7t} \in \mathbb{R}^n$, where $t = \lfloor \frac{n}{2^3} \rfloor$, then $d_L(x, c_0) = n + 4t$, $d_L(x, c_1) = 2n - 4t$, $d_L(x, c_2) = n+4t$, $d_L(x, c_3) = 4n-20t$, $d_L(x, c_4) = n+4t$, $d_L(x, c_5) = n+4t$, $d_L(x, c_6) = n+4t$ and $d_L(x, c_7) = n+4t$. Thus $r_L(C_I) \ge \min\{n + 4t, 2n - 4t, 4n - 20t\} = n + 4t \ge \frac{3n}{2}$. Thus $r_L(C_I) = \frac{3n}{2}$.

Let $x = \overbrace{u^2u^2\cdots u^2}^{\frac{n}{2}}\overbrace{000\cdots 0}^{\frac{n}{2}} \in \mathbb{R}^n$. The code $C_{II} = \{\alpha(u^2u^2\cdots u^2) \mid \alpha \in \mathbb{R}^n\}$. Then, $r_L(C_{II}) \ge 2n$. If, $x$ be any word in $\mathbb{R}^n$. Therefore, $r_L(C_{II}) \le 2n$ and hence, $r_L(C_{II}) = 2n$.

For $x = \overbrace{uu\cdots u}^{\frac{n}{2}}\overbrace{00\cdots 0}^{\frac{n}{2}} \in \mathbb{R}^n$ and the code $c_i \in \{\alpha(C_{III})|\alpha \in \mathbb{R}\}$ generated by $[uu\cdots u]$ is an $(n, 4, 2n)$ code. Thus, by definition of covering radius $r_L(C_{III}) \ge n$. Let $x$ be any word in $\mathbb{R}^n$. Then $r_L(C_{III}) \le 2n$ and hence, $n \le r_L(C_{III}) \le 2n$.

□

**Theorem 3.2.** *1.* $r_{GL}(C_I) = 2n$, *2.* $r_{GL}(C_{II}) = 2n$ *and*

*3.* $n \leq r_{GL}(C_{III}) \leq 2n$.

*Proof.* The proof follows from the Theorem 3.1, by using the generator matrices $G_I, G_{II}$ and $G_{III}$ with Generalized Lee Weight. □

**Theorem 3.3.** *1.* $r_{CE}(C_I) = 2n$,

*2.* $r_{CE}(C_{II}) = 2n$ *and*

*3.* $r_{CE}(C_{III}) = n$.

*Proof.* The proof is the same as the proof of the Theorem 3.1 and the generator matrices $G_I, G_{II}$ and $G_{III}$ with Chinese Euclidean Weight. □

# 4 Same Size of Blocks Repetition Code

Let $G_1 = [\overbrace{11\cdots 1}^{n}\overbrace{vv\cdots v}^{n}\overbrace{v^2v^2\cdots v^2}^{n}\overbrace{v^3v^3\cdots v^3}^{n}]$ be a generated matrix for the four block repetition code each of size $n$. The parameters of repetition code $BRep^{4n}$ : $[4n, 1, 4n, 4n, 8n, 8n]$. Using the generated matrix $G_1$, obtain

**Theorem 4.1.** *Let $C$ be a code over $\mathbb{R}$ generated by the matrix $G_1$, then $r_L(BRep^{4n}) = 6n, r_{GL}(BRep^{4n}) = 8n$ and $r_{CE}(BRep^{4n}) = 8n$.*

*Proof.* In Theorem 3.3 and [13] and the given generator matrix $G_1$, we get

$$r_L(BRep^{4n}) \geq 6n \tag{1}$$

Let $x = (u_1 \mid u_2 \mid u_3 \mid u_4) \in \mathbb{R}^{4n}$ where $u_1, u_2, u_3, u_4 \in \mathbb{R}^n$. Let us take in $u_1$, 0 appears $r_0$ times, 1 appears $r_1$ times, 2 appears $r_2$ times 3 appears $r_3$ times 4 appears $r_4$ times, 5 appears $r_5$times, 6 appears $r_6$times and 7 appears $r_7$times, in $u_2$, 0 appears $s_0$ times, 1 appears $s_1$ times, 2 appears $s_2$ times, 3 appears $s_3$ times, 4 appears $s_4$ times, 5 appears $s_5$ times, 6 appears $s_6$ times and 7 appears $s_7$ times, in $u_3$, 0 appears $t_0$ times, 1 appears $t_1$ times, 2 appears $t_2$ times, 3 appears $t_3$ times, 4 appears $t_4$ times, 5 appears $t_5$ times, 6 appears $t_6$ times and 7 appears $t_7$ times, in $u_4$, 0 appears $v_0$ times, 1 appears $v_1$ times, 2 appears $v_2$ times, 3 appears $v_3$ times, 4 appears $v_4$ times, 5 appears $v_5$ times, 6 appears $v_6$ times and 7 appear $v_7$ times, with $\sum_{i=0}^{7} r_i = \sum_{i=0}^{7} s_i = n = \sum_{i=0}^{7} t_i = \sum_{i=0}^{7} v_i$ and $c_i \in \{\alpha(G_1) | \alpha \in \mathbb{R}\}$. Then

$$d_L(x, c_0) = 4n - r_0 + r_2 + 3r_4 + r_6 - s_0 + s_2 + 3s_4 + s_6 - t_0 + s_t + 3t_4 + t_6 - v_0 + v_2 + 3v_4 + v_6,$$

$$d_L(x, c_1) = 4n - r_1 + r_3 + 3r_5 + r_7 - s_3 + s_5 + 3s_7 + s_1 - t_5 + t_7 + 3t_1 + t_3 - v_7 + v_1 + 3v_3 + v_5,$$

$$d_L(x, c_2) = 4n - r_2 + r_0 + r_4 + 3r_6 - s_6 + s_0 + 3s_2 + s_4 - t_2 + t_0 + t_4 + 3t_6 - v_6 + v_0 + 3v_2 + v_4,$$

$$d_L(x, c_3) = 4n - r_3 + r_5 + 3r_7 + r_1 - s_1 + s_3 + 3s_5 + s_7 - t_7 + t_1 + 3t_3 + t_5 - v_5 + v_7 + 3v_1 + v_3,$$

$$d_L(x, c_4) = 4n - r_4 + 3r_0 + r_2 + r_6 - s_4 + 3s_0 + s_2 + s_6 - t_4 + 3t_0 + t_2 + t_6 - v_4 + 3v_0 + v_2 + v_6,$$

$$d_L(x, c_5) = 4n - r_5 + r_7 + 3r_1 + r_3 - s_7 + s_1 + 3s_3 + s_5 - t_1 + t_3 + 3t_5 + t_7 - v_3 + v_5 + 3v_7 + v_1,$$

$$d_L(x, c_6) = 4n - r_6 + r_0 + 3r_2 + r_4 - s_2 + s_0 + s_4 + 3s_6 - t_6 + t_0 + 3t_2 + t_4 - v_2 + v_0 + v_4 + 3v_6,$$

$$d_L(x, c_7) = 4n - r_7 + r_1 + 3r_3 + r_5 - s_5 + s_7 + 3s_1 + s_3 - t_3 + t_5 + 3t_7 + t_1 - v_1 + v_3 + 3v_5 + v_7.$$

Therefore, $d_L(x, BRep^{4n}) = \min\{d_L(x, c_0), d_L(x, c_1), d_L(x, c_2), d_L(x, c_3), d_L(x, c_4), d_L(x, c_5), d_L(x, c_6), d_L(x, c_7)\}$ is less than or equal to $6n$.

Then $d_L(x, BRep^{4n}) \leq 6n$ and hence

$$r_L(BRep^{4n}) \leq 6n \tag{2}$$

By (1) and (2), then $r_L(BRep^{4n}) = 6n$.

Similarly,
$r_{GL}(BRep^{4n}) = 8n$ and $r_{CE}(BRep^{4n}) = 8n$.

□

The three-block repetition code $BRep^{3n} : (3n, 4, 2n, 6n, 8n)$ generated by
$G_2 = [\overbrace{uu\cdots u}^{n}\overbrace{u^2u^2\cdots u^2}^{n}\overbrace{uv\ uv\cdots uv}^{n}]$.

**Theorem 4.2.** *Let $C$ be a code over $\mathbb{R}$ generated by the matrix $G_2$. Then $r_L(BRep^{3n}) = 6n, r_{GL}(BRep^{3n}) = 6n$ and $r_{CE}(BRep^{3n}) = 4n$.*

*Proof.* Using, Theorem 3.3, and [13] and the given generator matrix $G_2$, the proof follows. □

**Corollary 1.** *Let $C$ be a code over $\mathbb{R}$. Then*

1. $G = [\overbrace{11\cdots 1}^{n}\overbrace{u^2u^2\cdots u^2}^{n}]$, *then* $r_L(BRep^{2n}) = \frac{7n}{2}, r_{GL}(BRep^{2n}) = 4n$ *and* $r_{CE}(BRep^{2n}) = 4n$.
2. $G = [\overbrace{11\cdots 1}^{n}\overbrace{uu\cdots u}^{n}]$, *then* $r_L(BRep^{2n}) = \frac{7n}{2}, r_{GL}(BRep^{2n}) = 4n$ *and* $r_{CE}(BRep^{2n}) = 3n$.
3. $G = [\overbrace{11\cdots 1}^{n}\overbrace{uu\cdots u}^{n}\overbrace{u^2u^2\cdots u^2}^{n}]$, *then* $r_L(BRep^{3n}) = \frac{11n}{2}, r_{GL}(BRep^{3n}) = 6n$ *and* $r_{CE}(BRep^{3n}) = 5n$.
4. $G = [\overbrace{11\cdots 1}^{n}\overbrace{vv\cdots v}^{n}\overbrace{v^2v^2\cdots v^2}^{n}\overbrace{v^3v^3\cdots v^3}^{n}]$, *then* $r_L(BRep^{4n}) = 6n, r_{GL}(BRep^{4n}) = 8n$ *and* $r_{CE}(BRep^{4n}) = 8n$.
5. $G = [\overbrace{uu\cdots u}^{n}\overbrace{u^2u^2\cdots u^2}^{n}\overbrace{uv\ uv\cdots uv}^{n}]$, *then* $r_L(BRep^{3n}) = 6n, r_{GL}(BRep^{3n}) = 6n$ *and* $r_{CE}(BRep^{3n}) = 4n$.
6. $G = [\overbrace{11\cdots 1}^{n}\overbrace{uu\cdots u}^{n}\overbrace{vv\cdots v}^{n}\overbrace{u^2u^2\cdots u^2}^{n}\overbrace{v^2v^2\cdots v^2}^{n}\overbrace{uv\ uv\cdots uv}^{n}\overbrace{v^3v^3\cdots v^3}^{n}]$, *then* $r_L(BRep^{7n}) = 12n$, $r_{GL}(BRep^{7n}) = 14n$ *and* $r_{CE}(BRep^{2n}) = 12n$.

*Proof.* The Proof follows from Theorem 3.1, Theorem 3.2 and Theorem 3.3. □

# 5 Different Size of Blocks Repetition Code

Two different size of block repetition code are defined as $\mathbb{R}$ (two different blocks of size $m$ and $n$ respectively): $BRep^{m+n} : [m+n, 1, \min\{m, m+n\}, \min\{2m, 2m+2n\}, \min\{4m, 3m+3n\}, \min\{2m, 2m+2n\}]$ generated by $G_3 = [\overbrace{11\cdots 1}^{m}\overbrace{u^2u^2\cdots u^2}^{n}]$.
In Corollary 1 can be easily generalized to two different length, using similar arguments to the following

**Theorem 5.1.** *Let $C$ be a code over $\mathbb{R}$ generated by the matrix $G_3$, then $r_L(BRep^{m+n}) = \frac{3m}{2} + 2n$, $r_{GL}(BRep^{m+n}) = 2m + 2n$ and $r_{CE}(BRep^{m+n}) = 2m + 2n$.*

In four different blocks of repetition code of size $m_1$, $m_2$, $m_3$ and $m_4$ in $\mathbb{R}$, is $BRep^{m_1+m_2+m_3+m_4}$ : $[m_1+m_2+m_3+m_4, 1, \{(m_1+m_2+m_3+m_4), \min\{(m_1+m_2+m_3+m_4), 2(m_1+m_2+m_3+m_4)\}, \min\{2(m_1+m_2+m_3+m_4), 4(m_1+m_2+m_3+m_4)\}\}]$ generated by

$$G_4 = [\overbrace{11\cdots 1}^{m_1}\overbrace{vv\cdots v}^{m_2}\overbrace{v^2v^2\cdots v^2}^{m_3}\overbrace{v^3v^3\cdots v^3}^{m_4}].$$

**Theorem 5.2.** *Let $C$ be a code and $G_4$ be a generator matrix of $C$ in $\mathbb{R}$, so*

$$r_L(BRep^{m_1+m_2+m_3+m_4}) = \frac{3}{2}(m_1+m_2+m_3+m_4),$$

$$r_{GL}(BRep^{m_1+m_2+m_3+m_4}) = 2(m_1+m_2+m_3+m_4),$$

$$r_{CE}(BRep^{m_1+m_2+m_3+m_4}) = 2(m_1+m_2+m_3+m_4).$$

*Proof.* Using Theorem 5.1, it can be obtained. □

**Corollary 2.** *Let $C$ be a code over $\mathbb{R}$ generated by the following generator matrices with covering radius*

1. $G = [\overbrace{11\cdots 1}^{m_1}\overbrace{uu\cdots u}^{m_2}]$, *then* $r_L(BRep^{m_1+m_2}) = \frac{3m_1}{2} + 2m_2$, $r_{GL}(BRep^{m_1+m_2}) = 2m_1 + 2m_2$ *and* $r_{CE}(BRep^{m_1+m_2}) = 2m_1 + m_2$.

2. $G = [\overbrace{11\cdots 1}^{m_1}\overbrace{uu\cdots u}^{m_2}\overbrace{u^2u^2\cdots u^2}^{m_3}]$, *then* $r_L(BRep^{m_1+m_2+m_3}) = \frac{3m_1}{2} + 2(m_2+m_3)$, $r_{GL}(BRep^{m_1+m_2+m_3}) = 2(m_1+m_2+m_3)$ *and* $r_{CE}(BRep^{m_1+m_2+m_3}) = 2(m_1+m_2) + m_3$.

3. $G = [\overbrace{uu\cdots u}^{m_1}\overbrace{u^2u^2\cdots u^2}^{m_2}\overbrace{uv\ uv\cdots uv}^{m_3}]$, *then* $r_L(BRep^{m_1+m_2+m_3}) = 2(m_1+m_2+m_3)$, $r_{GL}(BRep^{m_1+m_2+m_3}) = 2(m_1+m_2+m_3)$ *and* $r_{CE}(BRep^{m_1+m_2+m_3}) = m_1 + 2m_2 + m_3$.

4. $G = [\overbrace{11\cdots 1}^{m_1}\overbrace{uu\cdots u}^{m_2}\overbrace{vv\cdots v}^{m_3}\overbrace{u^2u^2\cdots u^2}^{m_4}\overbrace{v^2v^2\cdots v^2}^{m_5}\overbrace{uv\ uv\cdots uv}^{m_6}\overbrace{v^3v^3\cdots v^3}^{m_7}]$,
   *then* $r_L(BRep^{\sum_{i=1}^{7} m_i}) = 6(m_1+m_3+m_5+m_7) + 2m_4 + 4(m_2+m_6)$,
   $r_{GL}(BRep^{\sum_{i=1}^{7} m_i}) = 8(m_1+m_3+m_5+m_7) + 2m_4 + 4(m_2+m_6)$ *and*
   $r_{CE}(BRep^{\sum_{i=1}^{7} m_i}) = 8(m_1+m_3+m_5+m_7) + 2m_4 + 2(m_2+m_6)$.

*Proof.* Using Theorem 5.1 and 5.2. □

# 6 Conclusion

This work is for a finite ring with eight elements, that is the constructing new codes by concatenation are $\mathbb{Z}_2\mathbb{Z}_4$ codes. The estimation of the lower bound and upper bound for each block repetition code in $\mathbb{Z}_2\mathbb{Z}_4$ by using different distance and also different types of length, same type of length. These codes can be applied to complex situations encountered in all engineering fields. In future, I will extend this work to mix finite rings $\mathbb{Z}_m\mathbb{Z}_n(m > 2, n > 4)$ with all distance.

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