Journal of Nepal Mathematical Society (JNMS)
ISSN: 2616-0153 (Print), 2616-0161 (Online)
Vol. 8, Issue 1, 2025 (June): 30-38
DOI: https://doi.org/10.3126/jnms.v8i1.80311

Research Article
Received Date: March 18, 2025
Accepted Date: May 11, 2025
Published Date: June 25, 2025

On the Fibonacci and the Generalized Fibonacci Sequence

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Abstract: *Fibonacci numbers and their sequence are found abundantly in nature. There is a close relation among the Golden, Fibonacci, and Lucas ratios. Such ratios are inherent to design, architecture, construction, and even to the beauty of different natural and manmade solid objects. Other variants, like the k-generalized Fibonacci sequence, the Fibonacci p-numbers, Lucas numbers, and the ordinary Fibonacci numbers, have some interesting properties and are based on recurrent relations. This article describes the different structures, mathematical beauty, and identities related to such sequences and numbers. Some important applications of Fibonacci sequences are also explored.*

Keywords: Fibonacci numbers, Golden ratio, K-generalized Fibonacci sequences, Lucas numbers, Recurrence relations

1 Introduction

Italian mathematician Leonardo Pisano introduced the concept of the Fibonacci number, also known as Fibonacci from his nickname. Fibonacci considered the sequence to be an answer to questions like: How many pairs of rabbits will be produced in a year provided each rabbit pair bears a new pair every month, which becomes productive from the second month on?' There will be only one pair in the first and second months, but there will be two, three and five rabbit pairs in the third, fourth, and fifth months, and so on, resulting in a sequence of 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144.

Let's begin our discussion with an introduction to the golden ratio. Assume two positive numbers, x and y, satisfying $x > y$ inequality. Then their ratio, $x : y$, denoted by a Greek number ϕ is said to be in the golden ratio if the ratio of the larger to the smaller numbers equals the ratio of their sum to the larger number. Therefore, $\frac{x}{y} = \frac{x+y}{x}$. Then, $\phi = 1 + \frac{1}{\phi}$. It gives, $\phi^2 - \phi - 1 = 0$, a quadratic equation in ϕ having $\phi = \frac{1+\sqrt{5}}{2} = 1.61803398874989 48482...$, which is is also equal to $2 \times \sin 54^o$. Such an irrational number close to 1.6180 is the golden ratio. The golden ratio is also called the divine ratio. Such a ratio was studied as early as 300 BC by Euclid. It was observed in the solar system, geology, physics, and life sciences. There is a close relation of such ratio ϕ with the Fibonacci ratio. Before this, let us have a close look at the Fibonacci sequence with $t_n \leq 1000$ as 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987. Respective Fibonacci ratios are $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{8}{5}$, $\frac{13}{8}$, $\frac{21}{13}$, $\frac{34}{21}$, $\frac{55}{34}$, $\frac{89}{55}$, $\frac{144}{89}$, $\frac{233}{144}$, $\frac{377}{233}$, $\frac{610}{377}$, $\frac{987}{610}$, i.e., 1.000000, 2.000000, 1.500000, 1.666667, 1.666667, 1.625000, 1.615385, 1.619048, 1.617647, 1.618182, 1.617978, 1.618056, 1.618026, 1.618037, 1.6180379, approximately the same as the golden ratio ϕ. Hence, we get,

$$\phi = \lim_{n\to\infty} \frac{t_n}{t_{n-1}}.$$

We refer to the famous text like Gorbani [6] and Hemenway [7] and the references therein for more details about golden ratio ϕ, Fibonacci sequence, and the relationship between them along with some interesting mathematical properties.

This paper provides an eagle's-eye view of the Fibonacci sequence, Lucas's sequence, the golden ratio, and the K-generalized Fibonacci sequence, including some of their relations, identities, properties, and applications. Providing broad mathematical insight into the Fibonacci and the Fibonacci-type sequences, we have explored different recursion relations with their possible illustrative examples and opened the broader horizon for further research related to Fibonacci-type sequences. In Section 2, we present a quick literature review. The basics of the Fibonacci sequence is provided in Section 3. A discussion on the generalized Fibonacci sequence is presented in Section 4. Applications of Fibonacci are discussed in Section 5. Conclusion is presented in the Section 6.

2 Literature Review

A similar sequence of numbers was already known in the Indian subcontinent by Sanskrit grammarian Pingala much earlier in either name. Fibonacci numbers and their sequence are found abundantly in nature. It can also be used to define a spiral and is of interest to other fields, like biology and physics. They are frequently observed in nature in different natural objects like branching patterns of trees and leaves and the distribution of seeds in raspberries. The ratio of any number in the sequence to the number it followed (except the first few) are close to and related to the golden ratio, making the Fibonacci numbers a nature's number system. These numbers are of great use in various fields, such as architecture, engineering, technology, art, modern design, software engineering, and computing. Readers may find the study presented by Koshy [8], Livio [10], and Posamentier and Lehman [12] worthy of reviewing.

The Lucas numbers have a strong connection with Fibonacci numbers and follow a similar relation defined as $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$, with specific initial conditions of $L_1 = 1$ and $L_2 = 3$. Hence, the first 10 Lucas numbers are 1, 3, 4, 7, 11, 18, 29, 47, 76, 123. We can easily establish different relations between these two numbers, like:

1. $L_n \; = \; F_{n-1} + F_{n+1}$.
2. $F_n \; = \; \frac{1}{5}\left[L_n + L_{n-1}\right]$.
3. $\phi^n \; = \; \frac{L_n+\sqrt{5}F_n}{2}$.
4. $F_{2n} \; = \; L_n \cdot F_n$.
5. $L_n^2 \; = \; L_{n-1} \cdot L_{n+1} + 5\,(-1)^n, \;\; \forall n \geq 1$.
6. $L_n^2 \; = \; 5\,F_n^2 + 4(-1)^n$.
7. $2F_{m+n} \; = \; F_m \cdot L_n + F_n \cdot L_m$.
8. $L_{n+1} \cdot L_n \; = \; 5\,F_{n+1} \cdot F_n + 2(-1)^n$.

Various classical works on Fibonacci and the Lucas Numbers explore their fundamental properties, mathematical deductions, recurrence relations and their applications. Different researchers in different decades continue to investigate the generation, expositions, and applications in various field and inspire the mathematical exploration. The Fibonacci and the Lucas number and some other similar structures and their various properties and relations can be found in Koshy [8].

Miska et al. [11] introduced a new family of number sequence as the binary sequence to meet the Fibonacci, whereas Yaying et al. [14] investigated some of the geometric properties like approximation property, Hann-Banach extension property, in different spaces.The tricomplex Fibonacci numbers, their properties like recurrence relation, summation formula, and generating functions with some classical identities, such as Binet's formula and Cassini's identity, are also explored as an extension of the ordinary Fibonacci sequence by Costa et al. [4]. The distribution generated by a random in-homogeneous Fibonacci sequence with the respective probability distribution is also presented in Lipati and Szalay [9].

Fibonacci sequence and its retracements, fans, time zones, arcs etc. are also in stock market to analyze and predict market trends and movements. It is one of the fascinating topics in fractal geometry and also in Julia sets. Various programming and simulation techniques, such as Fibonacci search, heaps, and cubes, are utilized in search algorithms, and data structures as mentioned in Sharma et al. [13].

3 Fibonacci Sequence

Before starting the mathematical overview of Fibonacci, let us imagine a biologically idealistic scenario in which the population growth of rabbit pairs is considered. Assume a newly born pair of rabbits of opposite sexes is in a field, each pair in the field breeds, each breeding pair reaches sexual maturity, always mates once they become one month old and each end of month (EOM) dating, and consistently produces another pair of offspring, and they don't die but continually breed. So, the question is: how many pairs of rabbits

will there be at the end of the year?
The scenario looks like the following:

- There is a pair of rabbits in the field for entire 1st month.
- The rabbit pairs mate at the 1st month's EOM dating, and there will still be a pair of rabbits in the field.
- The rabbits will produce a new pair at the 2nd EOM dating and there will be 2 pairs of rabbits for entire 3rd month.
- The original pair will produce a 2nd pair at 3rd EOM dating such that there will be 3 rabbit pairs.
- At the 4th EOM dating, the earliest pair and the new pairs born two months ago will produce their offspring rabbit pairs resulting 5 pairs rabbit in total and so on.
- The rabbit pairs can be expressed as 1, 1, 2, 3, 5, 8,13, 21, 34, 55, 89, and 144 within a year, as in the form of a Fibonacci sequence.

Fibonacci numbers F_n follow the following recurrence relation,

$$F_{n+1} = F_n + F_{n-1},$$

for $n \geq 1$. The first two F_n satisfy $F_0 = 0$ and $F_1 = 1$.

Example 1. *Consider $\{F_n\}_0^\infty$ by the recursive formula as,*

$$p_0 = 0, \quad p_1 = 1, \quad p_n = \begin{cases} a\, p_{n-1} + p_{n-2} & \textit{if } n \textit{ is even} \\ b\, p_{n-1} + p_{n-2} & \textit{if } n \textit{ is odd} \end{cases}$$

for $n \geq 2$.
If $a = b = 1$, it gives the Fibonacci sequence.
If $a = b = 2$, it gives the Pell's number.
If $a = b = k$, for K be any positive integer, it gives the k-Fibonacci sequence.

An interesting fact on the Fibonacci sequence is that it has a connection with a finite geometric sequence where the Fibonacci sequence's term is the sum of a finite geometric series with $(\frac{1+\sqrt{5}}{2})^{n-1}$ as the first term and $\frac{1-\sqrt{5}}{1+\sqrt{5}}$ as the common ratio. Hence, as in elementary mathematics, the nth term of the Fibonacci sequence can be written as,

$$F_n = \Big(\frac{1+\sqrt{5}}{2}\Big)^{n-1}\Big[\frac{1-(\frac{1-\sqrt{5}}{1+\sqrt{5}})^n}{1-\frac{1-\sqrt{5}}{1+\sqrt{5}}}\Big] = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}} = \frac{\phi^n - \psi^n}{\sqrt{5}}.$$

This relation is also named the Binet's formula, Yilmaz et al. [15].

Example 2. *Express any Fibonacci number as in form of Binet's formula.*

Let a Fibonacci number $f_{12} = 144$, as a dozen dozens. Then numerically, we get,

$$F_{12} = \frac{(\frac{1+\sqrt{5}}{2})^{12} - (\frac{1-\sqrt{5}}{2})^{12}}{\sqrt{5}} = \frac{\phi^{12} - \psi^{12}}{\sqrt{5}}.$$

Consider the golden ratio be $\phi = \frac{1+\sqrt{5}}{2}$ and and its conjugate be, $\psi = \frac{1-\sqrt{5}}{2}$. Such a conjugate is the golden ratio conjugate, which is the negative of the negative of the golden ratio. Then by simple algebraic calculations with their respective values, we can establish:

1. $\psi + \phi - 1 = 0.$
2. $\psi + \frac{1}{\phi} = 0.$
3. $\phi^2 = \phi + 1.$
4. $\psi^2 = -\psi + 1.$
5. $\phi^{n+1} = \phi^n + \phi^{n-1}.$

Theorem 3.1. *Based on Binet's formula, we have,* $L_n = \phi^n + \psi^n \ \forall \ n \geq 1.$

Proof: We proceed through the method of induction. Here, $L_1 = \phi + \psi = 0$, conforming that the expression is true for $n = 1$. Let it be true for $n = 2, 3, ..., k$. We need to prove that the expression is valid for $n = K + 1$ as well. Now,
$L_{k+1} = L_k + L_{k-1}$
$= \phi^k + \psi^k + \phi^{k-1} + \psi^{k-1}$
$= \phi^k + \frac{\phi^k}{\phi} + \psi^k + \frac{\psi^k}{\psi}$
$= \phi^k(1 + \frac{1}{\phi}) + \psi^k(1 + \frac{1}{\psi})$
$= \phi^k(1 - \psi) + \psi^k(1 - \phi)$
$= \phi^k(1 + \phi - 1) + \psi^k(1 + \psi - 1)$
$= \phi^k + \psi^k$,
i.e. the expression is valid for $n = k + 1$, confirming $L_n = \phi^n + \psi^n \ \forall n \geq 1$.

□

Consider the first 10 Fibonacci (F_i) numbers from its sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55. Then sum up its first 8 terms $= \sum_{i=1}^{8} F_i = 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 = 54 = 55 - 1 = F_{10} - 1$. Thus, one gets

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1.$$

Similarly, for Lucas numbers, we get,

$$\sum_{i=1}^{n} L_i = L_{n+2} - 3.$$

Using recursive relations on Fibonacci sequence, we can write,

$$\begin{aligned}
F_n \cdot F_{n+1} &= F_n \cdot (F_n + F_{n-1}) \\
&= F_n^2 + F_{n-1} \cdot F_n \\
&= F_n^2 + F_{n-1} \cdot (F_{n-1} + F_{n-2}) \\
&= F_n^2 + F_{n-1}^2 + F_{n-1} \cdot F_{n-2} \\
&\quad \cdots\cdots\cdots\cdots \\
&\quad \cdots\cdots\cdots\cdots \\
&= F_n^2 + F_{n-1}^2 + ...F_2^2 + F_1 \cdot F_2 \\
&= F_n^2 + F_{n-1}^2 + ... + F_2^2 + F_1^2 \\
&= \sum_{i=1}^{n} F_i^2
\end{aligned}$$

Likewise, in the case of Lucas sequence, we have

$$L_n \cdot L_{n+1} = \sum_{i=1}^{n} L_i^2 + 2.$$

Theorem 3.2. *Let F_n be the nth term of the Fibonacci sequence and ψ be the conjugate of the golden ratio ϕ , then $\psi^n = F_n \cdot \psi + F_{n-1}$*

Proof:
Let ϕ and ψ be the golden ratio and its conjugate, respectively, then, $\phi \cdot \psi = \frac{1+\sqrt{5}}{2} \cdot \frac{1-\sqrt{5}}{2} = -1$.

Moreover, $\psi^2 = (\frac{1+\sqrt{5}}{2})^2 = \frac{3-\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} + 1 = \psi + 1$. Likewise, $\frac{\psi^2}{\psi} = 1 + \frac{1}{\psi}$, i.e., $\psi = 1 + \frac{1}{\psi}$.

Now, by using the Binet's formula from above, we get,

$$\begin{aligned}
F_n \cdot \psi + F_{n-1} &= \frac{\phi^n - \psi^n}{\sqrt{5}}.\psi + \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}} \\
&= \frac{1}{\sqrt{5}}\Big[\phi^n(\psi + \frac{1}{\phi}) - \psi^{n+1} - \psi^{n-1}\Big] \\
&= \frac{1}{\sqrt{5}}\Big[\phi^n\Big(\frac{\phi.\psi + 1}{\phi}\Big) - \psi^n(\psi + \frac{1}{\psi})\Big] \\
&= \frac{1}{\sqrt{5}}\Big[\phi^n.0 - \psi^n(\psi^2 + \psi - 1)\Big] \\
&= \frac{1}{\sqrt{5}}\Big[-\phi^n(2\psi - 1)\Big] \\
&= \frac{1}{\sqrt{5}}\Big[-\psi^n(-\sqrt{5})\Big] \\
&= \psi^n.
\end{aligned}$$

Hence, $\psi^n = F_n.\psi + F_{n-1}$.

□

For more intuition, we refer to Posamentier and Lehman [12].

Some other numbers like tribonacci are also similar to the Fibonacci and is defined as, $F_{n+1} = F_n + F_{n-1} + F_{n-2}$, for $n \geq 2$ with the initial conditions $F_0 = 0$, $F_1 = 1$ and $F_2 = 1$. Hence, such a tribonacci sequence becomes, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 564, 927, ... Tetranacci, pentacci, hexanacci, heptacci, and the silmilar higher order types sequences are also developed in similar fashion in literature with similar properties and are based on the Fibonacci as noticed by Feinberg [5]

4 Generalized Fibonacci Sequence

For non-zero natural numbers n, k, the family of k-generalized Fibonacci sequence is given by,
$G_n^{(k)} = \frac{1}{5^{\frac{k}{2}}}[(a + b\alpha)\alpha^{m+1} - (a + b\alpha)\beta^{m+1}]^r[(a + b\alpha)\alpha^m - (a + b\alpha)\beta^m]^{k-r}$ where $n = m\,k + r$, $0 \leq r < k$ and m be a non-zero natural number. Moreover, for a=0 and b=1, we get $G_n^{(k)} = F_n^{(k)}$. Likewise, for $k = 1$ and $r = 0$, we get $G_n^{(1)} = G_n$. One can express the relationship between the family of generalized and k-generalized Fibonacci numbers as, $G_n^{(k)} = (G_m)^{k-r}(G_{m+1})^r$ for $n = mk + r$. Based on such generalization, various recurrence relations and identities are established related to the k-Fibonacci sequence. For details, we refer to Yilmaz et al. [15].

For $a, b \in \mathbb{R}$ and $n \geq 1$, the Fibonacci numbers can also be generalized as, $G_{n+1} = G_n + G_{n-1}$ where $G_n = a$. The Fibonacci recurrence formula $F_{n+1} = F_n + F_{n-1}$ can be expressed in matrix form as the Fibonacci matrix like:

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}, \text{ n=1,2,...}$$

Thus, if we set, $y_{n+1} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, n=1,2,3,..., it gives, $y_{n+1} = F^n y_n$ for n=0, 1, 2, ...

By applying such recurrence repeatedly, we get,
$y_{n+1} = F^n y_1 = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, n=0, 1,2,...

Example 3. *The Fibonacci matrix has one positive eigen value and such a positive eigen value is the golden ratio.*

The Fibonacci recurrence, $F_{n+1} = F_n + F_{n-1}$ *can be expressed as the Fibonacci matrix like:*

$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$, *n=1,2,...*
Then by applying the recurrence formula repeatedly, we get,
$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}.$
Here, the basic matrix for the Fibonacci is $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ *For its eigen value, it gives the characteristic equations as,*

$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \implies \lambda^2 - \lambda - 1 = 0.$

$$\therefore \lambda_1 = \frac{1+\sqrt{5}}{2} \text{ and } \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

The positive eigen value of the Fibonacci matrix is nothing other than the golden ratio ϕ.

Example 4. *Express the Binet's formula in the form of the eigen values of the Fibonacci matrix.*

As mentioned above, the Binet's formula is given by,

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}} = \frac{\phi^n - \psi^n}{\sqrt{5}}.$$

Then, clearly by using the result obtained in above example 2, we get,

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}} = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}}.$$

Where, the respective eigen values are given by,

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \text{ and } \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

Let $\{F_n^{(2)}\}$ be the 2nd-order Fibonacci sequence defined by

$$F_n^{(2)} = F_{n-1}^{(2)} + F_{n-2}^{(2)}, \quad F_1^{(2)} = 1, \quad F_2^{(2)} = 1, \quad n \geq 3.$$

Then the nth term can be expressed in the matrix form as

$$\begin{bmatrix} F_n^{(2)} \\ F_{n-1}^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} F_2^{(2)} \\ F_1^{(2)} \end{bmatrix}.$$

To enhance convenience, an extension of the matrix representation to generalized Fibonacci sequences is needed, which can be done using the matrix formulas for the nth term.

Let $\{F_n^{(3)}\}$ be the 3rd-order Fibonacci sequence defined by

$$F_n^{(3)} = F_{n-1}^{(3)} + F_{n-2}^{(3)} + F_{n-3}^{(3)}, \quad F_1^{(3)} = 1, \quad F_2^{(3)} = 1, \quad F_3^{(3)} = 2, \quad n \geq 4.$$

The n^{th} term of the sequence can be represented as

$$\begin{bmatrix} F_n^{(3)} \\ F_{n-1}^{(3)} \\ F_{n-2}^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-3} \begin{bmatrix} F_3^{(3)} \\ F_2^{(3)} \\ F_1^{(3)} \end{bmatrix}.$$

This can be extended to an arbitrary order. Now, we will consider generalized Fibonacci numbers of order r,

$$F_n^{(r)} = F_{n-1}^{(r)} + F_{n-2}^{(r)} + \cdots + F_{n-r}^{(r)}, \quad n \geq r+1,$$

with initial conditions $F_1^{(r)} = F_2^{(r)} = \cdots = F_r^{(r)} = 1$.

We can represent the n-th term of the generalized Fibonacci sequence $\{F_n^{(r)}\}$ by matrices as:

$$\begin{bmatrix} F_n^{(r)} \\ F_{n-1}^{(r)} \\ \vdots \\ F_{n-r+1}^{(r)} \end{bmatrix} = A^{n-r} \begin{bmatrix} F_r^{(r)} \\ F_{r-1}^{(r)} \\ \vdots \\ F_1^{(r)} \end{bmatrix},$$

where A is the $r \times r$ companion matrix given by:

$$A = [a_{ij}] = \begin{cases} 1 & \text{if } i = 1, \\ 1 & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

A brief discussion on Fibonacci p-numbers is in order. Russian mathematician A.P. Stakhov introduced this Fibonacci numbers, which satisfy the following relation: $F_p(n) = F_p(n-1) + F_p(n-p-1)$ for $n \geq p+2$ where $F_p(1) = F_p(2) = ... = F_p(P+1)$. The Fibonacci p -numbers, take the form of the ordinary Fibonacci numbers for $p = 0$. Moreover, the p-Fibonacci numbers can also be generated from the generalized Fibonacci matrix. For details, we refer to Ahmad Praser [2], and Koshy [8].

5 Applications for Fibonacci

The Fibonacci note is highly applied to music. On a scale, a dominant note is the 5th note of a major scale, and the 8th note is the 13th one comprising an octave. This provides an added instance of Fibonacci numbers in a key musical relationship, interestingly, $\frac{13}{8}$ that is 1.625, approximating the Fibonacci ratio or the golden ratio ϕ. Great composers of Western music, such as Beethoven, Mozart, and Wagner, intentionally changed the rhythm of their compositions using a Fibonacci sequence, as in Posamentier and Lehmann [12].

The golden ratio, a mathematical translation of nature's algorithm , is a lesson on aesthetic perfection, beauty, harmony, and the pleasure of music. This divine proportion, which is also used in the design of various musical instruments like the piano, violin, guitars, and high-quality speaker wire , is a sight to behold. As mentioned earlier, such a golden ratio has an insight beauty and is closely connected to the

Fibonacci sequence. We refer to Adhikari and Kattel [1] and the references therein for a detailed insight into the beauty of Fibonacci.

Fibonacci sequences appear in the branching of trees, the arrangement of leaves on the stem, the fruitlets of a pineapple, the flowering of an artichoke, the arrangement of pinecones, the seeds of sunflower, the spirals of shells, the curves of waves, the family tree of honeybees, etc. It can be observed beautifully in various biological and natural settings like leaf arrangements, the number of spirals on a cactus, or sunflowers' seedbeds, and even in different geological and geographical structures. The number of steep and gradual spirals up the side of the pinecone is almost a Fibonacci number. For example, some pine-cones have three gradual and five steep spirals, while others have eight gradual and thirteen steep spirals as illustrated in Posamentier and Lehmann [12].

There are three types of bees in a beehive: the queen bee, the male bees, and the female bees. The queen bee's role is to lay eggs, while male bees do not have a specific function in the hive. In contrast, female bees are responsible for most of the hive'swork Posamentier and Lehmann, [12]. Male bees get developed from unfertilized eggs, meaning they have only a mother and no father. On the other hand, female bees develop from fertilized eggs, so they have both a mother and a father. For example, one male bee has one mother, two grandparents, three great-grandparents, five great-great-grandparents, and eight great-great-great-grandparents. Interestingly, the number of bees in each preceding generation follows a Fibonacci sequence Hemenway [8].

Nowadays, the golden ratio and Fibonacci numbers are commonly used in designing logos, magazine covers, plastic surgery, and more, as mentioned by Akhtaruzzaman and Shafie [3]. One of the most beautiful tiling patterns is the Fibonacci tiling, which is created using the Fibonacci sequence.

Many flowers have petal arrangements that correspond to Fibonacci numbers. For example, the White Calla Lily, Euphorbia, Trillium, Columbine, Bloodroot, Black-Eyed Susan, Shasta Daisy, and Field Daisies have petals in the following quantities: 1, 2, 3, 5, 8, 13, 21, and 34, respectively, as mentioned by Akhtaruzzaman and Shafie, [3].

A haiku is a three-line poem that follows specific syllable patterns: five syllables in the first line, seven in the second, and five in the third. Building on this structure, the Fibonacci poem, often called a Fib, is composed with a syllable pattern of 1, 1, 2, 3, 5, and 8 syllables in its lines Adhikari and Kattel [1].

Fibonacci retracements are widely used in the technical analysis of stock markets, whereas the Lucas sequences have applications in risk modeling and forecasting. Fibonacci numbers appear in quantum mechanics, like in quantized energy levels, and are also used in wave propagation models and signal processing. Whereas the Lucas sequences have been applied to vibration analysis and acoustics. Both of these numbers are connected to the Golden ratio and are of interest to researchers in various fields.

6 Conclusion

Fibonacci numbers and the Fibonacci sequence are found abundantly in nature and form nature's numbering systems. Such a sequence has a close relationship with the golden ratio and the Lucas number. Different variants, like the generalized k-Fibonacci sequence, Fibonacci p-numbers, Fibonacci matrix, the ordinary Fibonacci numbers, and the Lucas numbers have some interesting properties and are based on recurrent relations. Here, we describe the different identities, and some results related to such sequences and numbers. Some important recurrence relations with the applications of Fibonacci and k-Fibonacci sequences are also explored and illustrated with possible examples to open the broad horizon on such sequences and their similar types for further research.

Acknowledgments

The author wishes to thank the referees for their valuable time and the useful suggestions.

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