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Estimates of Some Dyadic Operators in the Weighted Setting

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Abstract: *The dyadic square function and the constant Haar multiplier have been estimated linearly with the A_2 characteristic of the weight, $[w]_{A_2}$ in the weighted Lebesgue space, $L^2(w)$. In this paper, we explore the estimation of the dyadic variable square function and the estimation of its composition with a constant Haar multiplier. This work shows that, the weight function, w in the dyadic reverse Hölder class 2, RH_2^d, characterizes the boundedness of S_w and $S_w \circ T_\sigma$. More precisely, our work is concerned with the boundedness of the dyadic variable square function and the boundedness of its composition from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, w)$; a single weight case.*

Keywords: Weight, Dyadic reverse Hölder class, Dyadic square function, Haar multiplier

1 Introduction

Dyadic operators are generated by Haar functions defined in dyadic intervals, $I \in \mathcal{D}$. Out of many dyadic operators, we focus only on the dyadic variable square function and the constant Haar multiplier. By an estimate of an operator in the weighted setting, we mean its boundedness in a weighted Lebesgue space $L^p(w), 1 < p < \infty$, related to the weight function w.

Estimation of operators in the weighted space(s) in terms of their weighted norms yields "Weighted Inequality". Basically, the main problem in the weighted inequality is to determine the existence of a constant "C" such that the inequality $(\int_X |Tf(x)|^q u(x)dx)^{\frac{1}{q}} \leq C(\int_X |Tf(x)|^p v(x)dx)^{\frac{1}{p}}$ governs for all $f \in L^p(X, dv)$, and a given operator T, where u, v are two measures on X, and $1 \leq p, q < \infty$. More precisely, the canonical problem is to determine the pair of weights u, v in the Euclidean space $X = \mathbb{R}^n$, and the existence of a constant "C" such that $(\int_{\mathbb{R}^n} |Tf(x)|^q u(x)dx)^{\frac{1}{q}} \leq C(\int_{\mathbb{R}^n} |Tf(x)|^p v(x)dx)^{\frac{1}{p}}$ for all $f \in L^p(\mathbb{R}^n, dv)$. The problems mentioned above can be categorized into two classes: the one-weight case, which involves working with a single weight $(u = v = w)$, and the two-weight case, which involves working with a pair (u, v) of weights. The one-weight case is a subcase of two-weight settings.

Muckenhoupt [10] was the first to solve the problem for the maximal function M in $\mathbb{R}^n$ for one weight case, and deduced the A_p condition, as the necessary and sufficient condition for its boundedness, although Helson and Szegö [5] have already in 1960, explored the necessary and sufficient condition for the boundedness of the Hilbert transform in the weighted $L^2(w)$ space, in a different way. Helson-Szegö theorem states that " the Hilbert transform H is bounded in $L^2(w)$ if and only if w can be expressed as, $w = \exp\{u + Hv\}, u, v \in L^\infty, \|u\|_\infty < \frac{\pi}{2}$". Hunt, Muckenhoupt, and Wheeden [7], in 1973, proved that $w \in A_p$ is the necessary and sufficient condition for the boundedness of the Hilbert transform in $L^p(w)$.

Linear dependency of the weighted norm of an operator to the Muckenhoupt A_p characteristic constant of the corresponding weight function was first shown by Buckley [1], in 1993 for the maximal function. It was the quantitative characterization. Wittwer [12], theorem 3.1, in 2000 showed that the norm of the martingale transform T_σ in the weighted $L^2(w)$ space depends linearly on the A_2 characteristic of the weight, $[w]_{A_2}$. Hukovic, Trail, and Volberg [6], in 2000 showed that the same linear bound also hold for the dyadic square function. Independently, Wittwer [13], in 2002 also proved the same result for the dyadic square function. One can see the chronology of the linear estimates on $L^2(w)$ for $w \in A_2$, page 13 in [11].

By including the variable $\frac{1}{\langle w \rangle_I^2} w(x)$, into the classical dyadic square function, we have introduced a variable square function, S_w given in the preliminaries, and proved that $w \in RH_2^d$ is the necessary and sufficient condition for the boundedness of the function. In addition, we proved that $w \in RH_2^d$ is also necessary and sufficient for the boundedness of $S_w \circ T_\sigma$, the composition of the two bounded operators.

The theory of weighted inequality is essential to many disciplines, including operator theory, the theory of quasi-conformal mapping, Fourier analysis, complex analysis, factorization theory, PDEs, and many more [11].

2 Preliminaries

In this section, we shall discuss some basic notations and definitions that are used in this paper.

2.1 Weight, A_p condition, maximal function, weighted norm, Dyadic reverse Hölder class RH_p^d

A locally integrable, non-negative, a.e. function, called a weight, is said to belong to $A_p, 1 < p < \infty$ if it satisfies the A_p condition:

$$[w]_{A_p} := \sup_Q \langle w\rangle_Q \langle (\frac{1}{w})^{\frac{1}{p-1}}\rangle_Q^{p-1} < \infty,$$

where the supremum is taken over all cubes having sides parallel to coordinate axes in $\mathbb{R}^n$, and $\langle f\rangle_Q := \frac{1}{|Q|}\int_Q f(x)dx$ denotes the average of f over Q, and $|Q|$ denotes the Lebesgue measure. The above quantity is also denoted by $\|w\|_{A_p}$, called the A_p characteristic constant of the weight. Obviously, for $p = 2$, the A_2 condition is $\sup_Q \langle w\rangle_Q \langle w^{-1}\rangle_Q < \infty$.

When $p = 1$, the class is called the A_1 class, and the weight w is said to be in the class, if there exists a positive constant c, satisfying $Mw \le cw$, almost everywhere, for the uncentered Hardy-Littlewood maximal function $M, Mf(x) := \sup_{x\in Q}\langle |f|\rangle_Q$. For the details of weight theory, one can follow [3, 4].

The norm of a function f in the weighted $L^p(\mathbb{R}^n, w), 1 < p < \infty$ is defined as:

$$\|f\|_{L^p(\mathbb{R}^n,w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{\frac{1}{p}} < \infty.$$

A weight w is in *dyadic reverse Hölder class p*, RH_p^d if there exists a constant $C > 0$ such that for every interval $I \in \mathcal{D}$,

$$\langle w\rangle_I^{-1}\langle w^p\rangle_I^{\frac{1}{p}} \le C, 1 < p < \infty,$$

where I is a dyadic interval of the standard dyadic grid, $\mathcal{D} := \{I = [j2^{-k}, (j+1)2^{-k}) : j, k \in \mathbb{Z}\}$. Thus, for $p = 2$, the condition becomes $\langle w\rangle_I^{-1}\langle w^2\rangle_I^{\frac{1}{2}} \le C$.

The smallest constant on the right is denoted by $[w]_{RH_p^d}$, called the RH_p^d- characteristic of the weight w, i.e., $[w]_{RH_p^d} := \sup_{I\in\mathcal{D}} \langle w\rangle_I^{-1}\langle w^p\rangle_I^{\frac{1}{p}} \le \infty$.

2.2 Haar multipliers: constant Haar multiplier, variable Haar multiplier [2, 8]

Formally, Haar multipliers are the Haar analogue of the pseudo-differential operators, replacing trigonometric functions by Haar functions given by

$$T_s f(x) := \sum_{I\in\mathcal{D}} s(x, I)\langle f, h_I\rangle h_I(x).$$

The variable symbol $s(x, I)$ is actually $\sigma_I \big(\frac{w(x)}{\langle w\rangle_I}\big)^t$, where $\{\sigma_I\}_{I\in\mathcal{D}}$ is a sequence of numbers, $\sigma_I = \pm 1$, $t \in \mathbb{R}$, and w is a weight. We may have the t-Haar multiplier as:

$$T^t_{w,\sigma} f(x) := \sum_{I\in\mathcal{D}} \sigma_I \left(\frac{w(x)}{\langle w\rangle_I}\right)^t \langle f, h_I\rangle h_I(x),$$

where $h_I(x)$ is the Haar function given by

$$h_I(x) = \frac{1}{|I|^{\frac{1}{2}}}\left(\mathbb{1}_{I_+} - \mathbb{1}_{I_-}\right)$$

corresponding to the dyadic interval I with its left and right child I_- and I_+ respectively, and $\mathbb{1}_I(x)$ is the characteristic function; $\mathbb{1}_I(x) = 1$ if $x \in I$, and zero otherwise. The set $\{h_I : I \in \mathcal{D}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, and an unconditional basis of $L^p(\mathbb{R}), 1 < p < \infty$ [11].
Basically, we may have two types of Haar multipliers:

- Constant Haar multiplier, $T_\sigma f(x) = \sum\limits_{I \in \mathcal{D}} \sigma_I \langle f, h_I \rangle h_I(x)$, $w \equiv 1$ & $s(x, I) = \sigma_I$ is independent of t, x also called the Martingale transform [11], and
- Variable Haar multiplier, $T_w^t f(x) = \sum\limits_{I \in \mathcal{D}} \frac{w(x)}{\langle w \rangle_I} \langle f, h_I \rangle h_I(x)$, $s(x, I)$ is dependent on x.

Note that, T_σ is bounded if and only if $\sigma = \{\sigma_I\}_{I \in \mathcal{D}}$ is bounded; Lemma 10 [8], and T_w^t is bounded if and only if $w \in RH_p^d$; Theorem 1 [8].

2.3 Dyadic square function, dyadic variable square function

Corresponding to a function f, the dyadic Littlewood-Paley square function is defined as:

$$Sf(x) := \left(\sum_{I \in \mathcal{D}} \frac{|\langle f, h_I \rangle|^2}{|I|} \mathbb{1}_I(x) \right)^{\frac{1}{2}},$$

where the summation is over all the dyadic intervals in $\mathcal{D}$.
For the function $f \in L^p, 1 < p < \infty, \|Sf\|_{L^p} \approx \|f\|_{L^p}$ with equality for $p = 2$, i.e., isometry in $L^2(\mathbb{R})$, namely $\|Sf\|_{L^2} = \|f\|_{L^2}$ [9, 11].

For a locally integrable function f, we have introduced the dyadic variable square function as:

$$S_w f(x) := \left(\sum_{I \in \mathcal{D}} \frac{w(x)}{\langle w \rangle_I^2} |\langle f, h_I \rangle|^2 \frac{\mathbb{1}_I(x)}{|I|} \right)^{\frac{1}{2}}.$$

3 Methods

There are many methods, such as the Bellman function method, Jones factorization theorem, Rubio de Francia's extrapolation theorem, Sparse operators, etc. to estimate the operators, but we use basic analysis tools to estimate the dyadic variable square function and its composition with the constant Haar multiplier.

4 Main Results

In this section, we have explored the boundedness of the dyadic variable square function, Theorem 1, and the boundedness of the composition of the dyadic variable square function with the constant Haar multiplier, Theorem 2.

Theorem 4.1. *The dyadic variable square function S_w is bounded from L^2 to $L^2(w)$ if and only if $w \in RH_2^d$.*

Proof. First assume that, $w \in RH_2^d$. Then, there exists a constant C such that $\langle w \rangle_I^{-1} \langle w^2 \rangle_I^{\frac{1}{2}} \leq C$ for each $I \in \mathcal{D}$, and we have

$$
\begin{aligned}
\|S_w f\|_{L^2(w)} &= \left(\int_{\mathbb{R}} \sum_{I\in\mathcal{D}} \frac{w(x)}{\langle w\rangle_I^2} |\langle f, h_I\rangle|^2 \frac{\mathbb{1}_I(x)}{|I|} w(x)dx\right)^{\frac{1}{2}} \\
&= \left(\sum_{I\in\mathcal{D}} \frac{|\langle f, h_I\rangle|^2}{\langle w\rangle_I^2} \frac{1}{|I|}\int_I w^2(x)dx\right)^{\frac{1}{2}} \\
&= \left(\sum_{I\in\mathcal{D}} |\langle f, h_I\rangle|^2 \langle w\rangle_I^{-2}\langle w^2\rangle_I\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{I\in\mathcal{D}} |\langle f, h_I\rangle|^2 C^2\right)^{\frac{1}{2}} \\
&= C\left(\sum_{I\in\mathcal{D}} |\langle f, h_I\rangle|^2\right)^{\frac{1}{2}}
\end{aligned}
$$

Using the fact that $\{h_I : I \in \mathcal{D}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, it follows that $\|S_w f\|_{L^2(w)} \leq C\|f\|_{L^2}$. This proves the boundedness of S_w from L^2 to $L^2(w)$.

Conversely, assume that $S_w : L^2 \to L^2(w)$ is bounded. Then, there exists a constant C such that $\|S_w(f)\|_{L^2(w)} \leq C\|f\|_{L^2}$, for all $f \in L^2$. In particular, taking $f = h_J$ for $J \in \mathcal{D}$, we have

$$\|S_w(h_J)\|_{L^2(w)} \leq C\|h_J\|_{L^2} \qquad \text{for all } J \in \mathcal{D}.$$

This implies

$$\left(\int_{\mathbb{R}} \sum_{I\in\mathcal{D}} \frac{w(x)}{\langle w\rangle_I^2} |\langle h_J, h_I\rangle|^2 \frac{\mathbb{1}_I(x)}{|I|} w(x)dx\right)^{\frac{1}{2}} \leq C\left(\int_{\mathbb{R}} h_J^2 dx\right)^{\frac{1}{2}}.$$

Since $\langle h_J, h_J\rangle = 1$, and $\langle h_I, h_J\rangle = 0$ for each $I \in \mathcal{D}, I \neq J$, the left hand side of the above inequality can be rewritten as

$$\left(\int_{\mathbb{R}} \frac{w(x)}{\langle w\rangle_J^2} \frac{\mathbb{1}_J(x)}{|J|} w(x)dx\right)^{\frac{1}{2}} = \left(\int_J \frac{w(x)}{\langle w\rangle_J^2} \frac{\mathbb{1}_J(x)}{|J|} w(x)dx\right)^{\frac{1}{2}} = \left(\langle w\rangle_J^{-2} \frac{1}{|J|}\int_J w^2(x)\, dx\right)^{\frac{1}{2}} = \langle w\rangle_J^{-1}\langle w^2\rangle_J^{\frac{1}{2}}.$$

Moreover, $\int_{\mathbb{R}} h_J^2(x)dx = \int_{\mathbb{R}} \frac{\mathbb{1}_J(x)}{|J|}dx = \frac{1}{|J|}\int_J \mathbb{1}_J(x)dx = 1.$
Combining these results, we thus have

$$\langle w\rangle_J^{-1}\langle w^2\rangle_J^{\frac{1}{2}} \leq C$$

for each $J \in \mathcal{D}$, which proves that $w \in RH_2^d$. □

Theorem 4.2. *Let w be a weight, and suppose T_σ be the Haar multiplier defined by $T_\sigma(f) = \sum_{I\in\mathcal{D}} \sigma_I\langle f, h_I\rangle h_I$, where $\sigma_I \in \{-1, 1\}$ for each $I \in \mathcal{D}$. Then $S_w \circ T_\sigma$ is bounded from L^2 to $L^2(w)$ if and only if $w \in RH_2^d$.*

Proof. First assume that, $w \in RH_2^d$. Then we may choose a constant $C > 0$ such that $\langle w\rangle_I^{-1}\langle w^2\rangle_I^{\frac{1}{2}} \leq C$.
Now, we have $\langle T_\sigma f, h_I\rangle = \langle \sum_{I\in\mathcal{D}} \sigma_I\langle f, h_I\rangle h_I, h_I\rangle = \sigma_I\langle f, h_I\rangle$.

Therefore, we have

$$\begin{aligned}\|S_w(T_\sigma f)\|_{L^2(w)} &= \left(\int_{\mathbb{R}} \sum_{I\in\mathcal{D}} \frac{w(x)}{\langle w\rangle_I^2} |\langle T_\sigma f, h_I\rangle|^2 \frac{\mathbb{1}_I(x)}{|I|} w(x)dx\right)^{\frac{1}{2}} \\ &= \left(\sum_{I\in\mathcal{D}} \frac{|\langle f, h_I\rangle|^2}{\langle w\rangle_I^2} \frac{1}{|I|}\int_I w^2(x)dx\right)^{\frac{1}{2}} ;\ |\sigma_I|^2 = 1 \\ &= \left(\sum_{I\in\mathcal{D}} |\langle f, h_I\rangle|^2 \langle w\rangle_I^{-2}\langle w^2\rangle_I\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{I\in\mathcal{D}} |\langle f, h_I\rangle|^2 C^2\right)^{\frac{1}{2}} \\ &= C\left(\sum_{I\in\mathcal{D}} |\langle f, h_I\rangle|^2\right)^{\frac{1}{2}}\end{aligned}$$

Since the Haar functions form an orthonormal basis for L^2, this gives

$$\|S_w(T_\sigma f)\|_{L^2(w)} \leq C\|f\|_{L^2}.$$

Conversely, assume that $S_w \circ T_\sigma : L^2 \to L^2(w)$ is bounded. Then, there exists a constant C such that

$$\|S_w(T_\sigma f)\|_{L^2(w)} \leq C\|f\|_{L^2}, \qquad \text{for all } f \in L^2.$$

Taking $f = h_J$, for $J \in \mathcal{D}$, we have

$$\|S_w(T_\sigma h_J)\|_{L^2(w)} \leq C\|h_J\|_{L^2}.$$

This implies

$$\left(\int_{\mathbb{R}} \sum_{I\in\mathcal{D}} \frac{w(x)}{\langle w\rangle_I^2} |\langle T_\sigma h_J, h_I\rangle|^2 \frac{\mathbb{1}_I(x)}{|I|} w(x)dx\right)^{\frac{1}{2}} \leq C.$$

Note that

$$\begin{aligned}\left(\int_{\mathbb{R}} \sum_{I\in\mathcal{D}} \frac{w(x)}{\langle w\rangle_I^2} |\langle T_\sigma h_J, h_I\rangle|^2 \frac{\mathbb{1}_I(x)}{|I|} w(x)dx\right)^{1/2} &= \left(\int_{\mathbb{R}} \frac{w(x)}{\langle w\rangle_J^2} |\sigma_J|^2 |\langle h_J, h_J\rangle|^2 \frac{\mathbb{1}_J(x)}{|J|} w(x)dx\right)^{1/2} \\ &= \left(\int_J \frac{w(x)}{\langle w\rangle_J^2} \frac{\mathbb{1}_J(x)}{|J|} w(x)dx\right)^{1/2} ;\ |\sigma_J|^2 = 1 \\ &= \left(\langle w\rangle_J^{-2} \frac{1}{|J|}\int_J w^2(x)dx\right)^{1/2} \\ &= \langle w\rangle_J^{-1}\langle w^2\rangle_J^{1/2}.\end{aligned}$$

Thus, we have $\langle w\rangle_J^{-1}\langle w^2\rangle_J^{\frac{1}{2}} \leq C$ for each $J \in \mathcal{D}$, proving that $w \in RH_2^d$. □

5 Conclusion

Linear dependence of the weighted norms of classical dyadic square function, and constant Haar multiplier in the weighted space, $L^2(w)$ to the A_p characteristic of the weight w, $[w]_{A_2}$ exists already, in literature. In this research, we have proved the dependency of the weighted norm of the dyadic variable square function in $L^2(w)$ to the dyadic reverse Hölder characteristic, $[w]_{RH_2^d}$ of the weight w. Moreover, we have also proved the similar result for the composition of the dyadic variable function with the constant Haar multiplier. We may also extend this idea to the variable Haar multiplier for the multilinear case.

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