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On Certain Properties of Numerical Radius Preservers in Nonunital Banach Algebras

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Abstract: *Numerical radius preserver problem is one of the linear preserver problems which have been studied over decades. However, this problem still remains interesting as it has not been solved in totality. Only partial solutions have been obtained in special cases like unital Banach algebras and C*-algebras among others. No solutions have been obtained in the nonunital cases. In this paper, we characterize certain properties of numerical radius preservers in nonunital Banach algebras. In particular, we show that every numerical radius preserver is surjective, continuous and real linear. Moreover, they preserve multiplicativity, commutativity, involution and is a* $*$*-isomorphism.*

Keywords: Numerical radius, Linear preserver, Nonuinital Banach algebra

1 Introduction

Linear preserver problems (LPP) are very interesting problems in functional analysis which have been studied over decades (see [1], [3]-[7] and [8] and the references therein). These include preservers of invertibility [2], orthogonality [4], numerical range [6], numerical radius (Nr) [13], norm[7], spectrum [9], spectral radius [11], positivity among others([12]-[19] and the references therein). Two aspects of the LPP which have been studied independently are considered mostly are orthogonality preservers and numerical radius preservers (NRP) [16].

The first notion of LPP which we consider in this study is that of NRP. This concept has also been given a lot of consideration over time(see [17]-[22]). For instance [20] characterized certain conditions that maps must satisfy in order for them to be preserving Nr particularly in C*-algebras. This was further extended by the work of [14] to include bilinear forms.

The work of [21] also considered linear maps that preserve particular aspects for instance the closure of NR on different types of algebras like the nest algebras having the property characterized with maximal atomic nest. In [17], the author gave a consideration to Nr preservers and the interest was keen on $B(H)$. However, [15], [24] and [25] in the same spirit discussed Nr distance preserving maps under certain restrictions on $B(H)$.

Now, we provide a review of works that have been done by various researchers on numerical radius preservers. We begin from the earlier years by the work of [19] on self adjoint operators.

Theorem 1.1 ([19])**.** *The adjoint T^* of $T \in B(H)$ preserves Nr of T.*

Theorem 1.1 shows that T^* preserves the numerical radius of T under a C^*-isomorphism in $B(H)$. It further stipulates that the self-adjointedness of T guarantees that the numerical radius and the usual operator norm coincides. However, this result is applicable only on special cases when we have the C^*-algebra with involution. There is need to establish if the conditions hold in nonunital Banach algebras. The work of [19] was extended by [5] as follows in the next theorem.

Theorem 1.2 ([5], Theorem 3)**.** *The operator $T \in B(H)$ preserves the numerical radius of T by a scalar modulo 1 under C^*-isomorphism.*

Theorem 1.2 also considered a special case and it is interesting to determine the the conditions in a general Banach space setting particularly in nonunital Banach algebras.

Remark 1.3. Earlier in the years, research was geared on matrices but recently numerical radius preservers are being considered on matrices and other operators [18].

We note that this work is an original work that contributes to functional analysis and in particular, operator theory. Moreover, our topic gives a very good generalizations of the concepts of LPP for example NR and Nr in Banach algebras. We give an introduction which entails a historical background of the past work done; Preliminaries that has basic definitions; Main results and, finally the conclusion in that order.

2 Preliminaries

Certain preliminary concepts are instrumental in this study. We give them under this section for ease of understanding of the work.

Definition 2.1 ([23], Section 2)**.** Linear preserver problems involves the study of maps that preserve certain properties or structures in linear spaces for example norm, numerical range, numerical radius, orthogonality, invertibility, angle among others.

Definition 2.2 ([11], Section 2)**.** Let $\mathcal{B}(H)$ be a Banach algebra. The numerical range of $A \in \mathcal{B}(H)$, where H is a Hilbert space, is defined by: $W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$ while the numerical radius of A is defined by $w(A) = \sup\{|z| : z \in W(A)\}$. The mapping $\phi \in B(H)$ is called a numerical radius preserver if $\phi(w(A)) = w(A)$. The set of all numerical radius preservers is denoted by $\mathcal{B}(H)_{NRP}$.

Definition 2.3 ([25], Definition 1.9)**.** A Banach algebra Ω that does not contain a multiplicative identity I is called a nonunital Banach algebra. That is there does not exist $I \in \Omega$ such that $AI = IA = A$, for all $A \in \Omega$.

3 Main Results

Certain properties of NRP are very instrumental in analysis. These include linearity, continuity among others. We begin with characterization of linearity of NRP. We note that $\mathcal{B}(\mathcal{H})^{NUBA}_{NRP}$ is the nonunital Banach algebra (NUBA) of all maps on a Hilbert space $\mathcal{H}$.

Proposition 3.1. *Let $\chi \in \mathcal{B}(\mathcal{H})^{NUBA}_{NRP}$ be an onto map. Then χ is real-linear.*

Proof. First, lets consider numerical radius in nonunital Banach algebras. For the unital case, the numerical radius of $\varpi \in \mathcal{B}(\mathcal{H})^{NUBA}_{NRP}$ is given as $w(\varpi) = \sup\left\{|\varrho(\varpi)| : \varrho \in S(\mathcal{B}(\mathcal{H})^{NUBA}_{NRP})\right\}$, where $S(\mathcal{B}(\mathcal{H})^{NUBA}_{NRP})^*$ forms the set of states on $\mathcal{B}(\mathcal{H})^{NUBA}_{NRP}$, that is, linear functionals $\varrho \in (\mathcal{B}(\mathcal{H})^{NUBA}_{NRP})^*$ with $\varrho(1) = 1$. Now, in the nonunital case, we consider the unitization $\widetilde{\mathcal{B}(\mathcal{H})}^{NUBA}_{NRP} = \mathcal{B}(\mathcal{H})^{NUBA}_{NRP} \oplus \mathbb{C}$, with multiplication $(\varpi, \lambda)(\xi, \mu) = (\varpi\xi + \lambda\xi + \mu\varpi, \lambda\mu)$, and we consider the extension of the norm. We then define the NR of $\varpi \in \mathcal{B}(\mathcal{H})_{NRP^{NUBA}}$ via the restriction of states on $\widetilde{\mathcal{B}(\mathcal{H})}^{NUBA}_{NRP}$. We know that NR is characterized by: Absolute homogeneity, that is, $w(\lambda\varpi) = |\lambda|w(\varpi)$ for all $\lambda \in \mathbb{C}$; Subadditivity-$w(\varpi+\xi) \leq w(\varpi)+w(\xi)$; and Nondegeneracy: $w(\varpi) = 0$ implies that $\varpi = 0$. Therefore, $w(\cdot)$ as a norm on $\mathcal{B}(\mathcal{H})^{NUBA}_{NRP}$ is well defined. Since χ preserves NR and is onto, we obtain that $w(\chi(\varpi) - \chi(\xi)) = w(\varpi - \xi)$, so ϕ is an normed isometry via $w(\cdot)$. Next we apply the Mazur–Ulam Theorem which states that any normed onto isometry χ with $\chi(0) = 0$ is real-linear. In fact, consider $\pi(\varpi) = \chi(\varpi) - \chi(0)$. Then π is also a normed onto isometry and satisfies $\pi(0) = 0$. By the Mazur–Ulam theorem, π is real-linear. Therefore, we have $\chi(a) = \pi(a) + \chi(0)$. To complete the proof, we prove that the shift term disappears. Suppose that $\chi(\varpi) = \pi(\varpi) + \omega$, for some fixed $\omega = \chi(0)$. Then $w(\chi(\varpi)) = w(\pi(\varpi) + \omega)$. By the triangle inequality we have that $w(\pi(\varpi) + \omega) \leq w(\pi(\varpi)) + w(\omega) = w(\varpi) + w(\omega)$, with equality only if $\pi(\varpi)$ and c satisfies linear dependence in a particular direction for all ϖ. This is a contradiction of the onto property of π unless $\omega = 0$. Hence, $\chi(0) = 0$, and so $\chi = \pi$ is real-linear. □

The next results shows that NRPs preserve Jordan product in NUBA.

Proposition 3.2. *Every onto $\chi \in \mathcal{B}(\mathcal{H})^{NUBA}_{NRP}$ preserves Jordan product.*

Proof. From Proposition 3.1, we know that χ is a real and linear normed onto isometry. Consider $\varpi, \xi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$, and also consider $\zeta = \varpi + \xi$ and $\eta = \varpi - \xi$. Let $\varpi = \frac{1}{2}(\zeta + \eta)$, $\xi = \frac{1}{2}(\zeta - \eta)$, which gives, $\varpi\xi + \xi\varpi = (\varpi + \xi)^2 - \varpi^2 - \xi^2 = \zeta^2 - \varpi^2 - \xi^2$, which defines algebraic structure of JP. By [13], real-linear isometries on BAs preserve JPs. Therefore, $\chi(\varpi \circ \xi) = \chi\left(\frac{1}{2}(\varpi\xi + \xi\varpi)\right) = \frac{1}{2}(\chi(\varpi\xi) + \chi(\xi\varpi)) = \frac{1}{2}(\chi(\varpi)\chi(\xi) + \chi(\xi)\chi(\varpi)) = \chi(\varpi) \circ \chi(\xi)$. □

The next results shows that NRPs preserve multiplicativity in NUBA.

Proposition 3.3. *Every onto $\chi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ preserves multiplicativity.*

Proof. By Proposition 3.2, χ preserves JP and so $\chi(\varpi^2) = \chi(\varpi \circ \varpi) = \chi(\varpi) \circ \chi(\varpi) = \chi(\varpi)^2$, for all $\varpi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$. Utilizing χ on both sides of the associative equation $\varpi\xi = \frac{1}{2}\left((\varpi + \xi)^2 - \varpi^2 - \xi^2\right)$, we have $\chi(\varpi\xi) = \frac{1}{2}\left(\chi((\varpi + \xi)^2) - \chi(\varpi^2) - \chi(\xi^2)\right)$ and invoking $\chi(\zeta^2) = \chi(\zeta)^2$, we obtain $\chi(\varpi\xi) = \frac{1}{2}\left((\chi(\varpi) + \chi(\xi))^2 - \chi(\varpi)^2 - \chi(\xi)^2\right) = \chi(\varpi)\chi(\xi) + \chi(\xi)\chi(\varpi)$. In a similar way, $\chi(\xi\varpi) = \chi(\xi)\chi(\varpi) + \chi(\varpi)\chi(\xi)$. So, $\chi(\varpi\xi) - \chi(\varpi\xi) = \chi(\varpi)\chi(\xi) - \chi(\xi)\chi(\varpi) = [\chi(\varpi), \chi(\xi)]$ and $\chi([\varpi, \xi]) = [\chi(\varpi), \chi(\xi)]$. Also, $\chi(\varpi \circ \xi) = \frac{1}{2}(\chi(\varpi)\chi(\xi) + \chi(\xi)\chi(\varpi))$. Hence, by Proposition 3.1 we have that χ is real linear and by Proposition 3.2 χ preserves JP. So, $\chi(\varpi\xi) = \chi(\varpi)\chi(\xi)$ which implies that χ is a multiplicative NRP which is linear and preserves JP.

□

In the next result, we consider the involutive property. We show that NRP also preserve involution in a NUBA which is a $*$-algebra.

Lemma 3.4. *Consider $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ having a canonical form with a *-identity. Then every onto $\chi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ preserves involution.*

Proof. Consider $\varpi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ as $\varpi = \varpi_1 + i\varpi_2$, where $\varpi_1 = \frac{1}{2}(\varpi + \varpi^*)$ and $\varpi_2 = \frac{1}{2i}(\varpi - \varpi^*)$ in $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ are self-adjoint. We need to show that $\chi(\varpi^*) = \chi(\varpi)^*$. Now, $\varpi^* = \varpi_1 - i\varpi_2$ and since χ is real-linear by Proposition 3.1, $\chi(\varpi) = \chi(\varpi_1 + i\varpi_2) = \chi(\varpi_1) + i\chi(\varpi_2)$, and $\chi(\varpi^*) = \chi(\varpi_1 - i\varpi_2) = \chi(\varpi_1) - i\chi(\varpi_2)$. It is enough that $\chi(\varpi_j)$ is self-adjoint for $j = 1, 2$. In deed let $\varpi = \varpi^*$. Since χ is NRP then by definition $w(\chi(\varpi)) = w(\varpi)$. But for ϖ, the $w(\varpi) = r(\varpi) = \|\varpi\|$. Suppose that $\chi(\varpi)$ is self-adjoint, then without loss of generality, $w(\chi(\varpi)) < \|\chi(\varpi)\|$. This contradicts the fact that $w(\varpi) = r(\varpi) = \|\varpi\|$. Therefore, $\chi(w(\varpi) = r(\varpi) = \|\varpi\|)$ must be self-adjoint. Hence, for any $\varpi = \varpi_1 + i\varpi_2$ with ϖ_1, ϖ_2 self-adjoint, we have that $\chi(a^*) = \chi(\varpi_1 - i\varpi_2) = \chi(\varpi_1) - i\chi(\varpi_2) = \chi(\varpi)^*$ which shows that χ preserves involution when $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ has a canonical form $\widehat{\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}}^*$ with a *-identity embedded in it.

□

A very important characterization of maps in NUBA involves isomorphisms. In the next theorem we show that we can obtain *-isomorphisms from NRPs.

Theorem 3.5. *Consider $(\mathcal{B}(\mathcal{H})_{NRP}^{NUBA})_1$ and $(\mathcal{B}(\mathcal{H})_{NRP}^{NUBA})_2$ both complex with *-identity. Then we have that $\chi : (\mathcal{B}(\mathcal{H})_{NRP}^{NUBA})_1 \to (\mathcal{B}(\mathcal{H})_{NRP}^{NUBA})_2$ is an onto real-linear *-isomorphism.*

Proof. First, we need to show the $1-1$ property. Let $\chi(\varpi) = \chi(\xi)$. This gives $\chi(\varpi - \xi) = 0$. But $w(\chi(\varpi - \xi)) = 0$, so it follows that $w(\varpi - \xi) = 0$. It is known from [10] that for Banach *-algebras, $w(\zeta) = 0 \Rightarrow \zeta = 0$, so $\varpi = \xi$. This shows that χ is $1-1$. By Lemma 3.4 χ is real-linear, multiplicative, and *-preserving meaning χ is a real-linear *-homomorphism. Next, we show that NRP has norm preservation on self-adjoint elements. To see this, let $\varpi = \varpi^*$ then $\|\varpi\| = w(\varpi) = w(\chi(\varpi)) = \|\chi(\varpi)\|$, which shows that χ is isometrically endowed on the self-adjoint part of $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$. We can also consider norm preservation in general. Now, $\varpi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ can be written as $\varpi = \varpi_1 + i\varpi_2$ with ϖ_1, ϖ_2 in the self-adjoint part of $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$. Therefore, $\chi(a) = \chi(\varpi_1) + i\chi(\varpi_2)$, with $\chi(\varpi_1), \chi(\varpi_2)$ self-adjoint. But $\|\chi(\varpi_j)\| = \|\varpi_j\|$, so we obtain $\|\chi(\varpi)\| \leq \|\chi(\varpi_1)\| + \|\chi(\varpi_2)\| = \|\varpi_1\| + \|\varpi_2\| \geq \|\varpi\|$. The reverse inequality, can be obtained similarly using χ^{-1} hence $\|\chi(\varpi)\| = \|\varpi\|$. This shows that χ is a *-isomorphism. □

The following consequence which states that NRP on self-adjoint elements establish *-isomorphisms follows immediately.

Corollary 3.6. *Consider a complex $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ with *-identity. Then we have that $\chi : \mathcal{B}(\mathcal{H})_{NRP}^{NUBA} \to \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ satisfies $w(\chi(\varpi)) = w(\varpi)$, for all $\varpi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$.*

Proof. Suppose that ϖ is self-adjoint. Then by Theorem 3.5, $\|\varpi\| = w(\varpi) = w(\chi(\varpi)) = \|\chi(\varpi)\|$, so χ preserves the norm on the self-adjoint part of $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$. But $\varpi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ decomposes as $\varpi = \varpi_1 + i\varpi_2$ with $\varpi_1, \varpi_2 \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ which is a self-adjoint algebra. Let $\chi(\varpi^*) = \chi(\varpi)^*$, then NR is preserved for any ϖ by $w(\varpi)^2 = w(\varpi_1)^2 + w(\varpi_2)^2 = w(\chi(\varpi_1))^2 + w(\chi(\varpi_2))^2 = w(\chi(\varpi))^2$. □

Since we are studying NRP in NUBA, it is obvious that identity is not preserved in $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$. However, this is true in the unital case. We give a counterexample in the unital case showing that NRPs are identity or reverse identity map preservers. Here, $\mathcal{B}(\mathcal{H})_{NRP}^{UBA}$ means unital Banach algebra of all maps on a Hilbert space $\mathcal{H}$.

Example 3.7. Consider $(\mathcal{B}(\mathcal{H})_{NRP}^{UBA})_1$ and $(\mathcal{B}(\mathcal{H})_{NRP}^{UBA})_2$ both complex with *-identity. Then we have that $\chi : (\mathcal{B}(\mathcal{H})_{NRP}^{UBA})_1 \to (\mathcal{B}(\mathcal{H})_{NRP}^{UBA})_2$ preserves the numerical radius.

To see this, by Corollary 3.6, χ preserves NR, so we have that $w(\chi(\varpi)) = w(\varpi), \quad$ for all $\varpi \in (\mathcal{B}(\mathcal{H})_{NRP}^{UBA})_1$, which implies that $\|\chi(\varpi)\| = \|\varpi\|$, for all $\varpi \in (\mathcal{B}(\mathcal{H})_{NRP}^{UBA})_1$ meaning is a homomorphism. It is now sufficient to establish different forms of χ. If $\chi(\varpi) = \varpi$, then $w(\chi(\varpi)) = w(\varpi)$, therefore χ is the identity map. But if $\chi(\varpi) = -\varpi$ then $w(\chi(\varpi)) = w(-\varpi) = w(\varpi)$, so χ is the reverse identity map.

Remark 3.8. From Example 3.7, it is clear that if NUBA are unitized then NRP are identity map preservers.

The next results is a general characterization which shows that onto NRPs are continuous and preserve commutativity in NUBA.

Theorem 3.9. *Every onto $\chi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ is continuous and preserves commutativity.*

Proof. We begin by showing that χ satisfies continuity criterion. Let (ϖ_n) in $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ be converging to $\varpi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$, so $(\chi(\varpi_n))$ converges to $\chi(\varpi)$ in $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$. But χ is NRP, so $\|\chi(\varpi)\| = w(\chi(\varpi)) = w(\varpi) = \|\varpi\|$, showing that χ is norm preserving in $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$. Let $\varpi_n \to \varpi$ in $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$. Then we have by isometrical property that $\|\chi(\varpi_n) - \chi(\varpi)\| = \|\chi(\varpi_n - \varpi)\| = \|\varpi_n - \varpi\|$. Therefore, $\|\chi(\varpi_n) - \chi(\varpi)\| = \|\varpi_n - \varpi\| \to 0$ as $\varpi_n \to \varpi$. proving that $\chi(\varpi_n) \to \chi(\varpi)$ in $\mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$. Hence, χ is continuous. Next we prove preservation of commutativity. Now for $\varpi\xi$, we have that $w(\chi(\varpi\xi)) = w(\varpi\xi)$. By the sub-multiplicativity of the NR, $w(\varpi\xi) \leq w(\varpi)w(\xi)$, which gives $w(\chi(\varpi\xi)) \leq w(\chi(\varpi))w(\chi(\xi))$. But χ is a NRP, so we have $w(\chi(\varpi)) = w(\varpi)$ and $w(\chi(\xi)) = w(\xi)$, which gives $w(\chi(\varpi\xi)) \leq w(\varpi)w(\xi)$. Now, $w(\chi(\varpi\xi)) = w(\varpi\xi)$, so $w(\varpi\xi) = w(\chi(\varpi\xi)) \leq w(\varpi)w(\xi)$. This leads to a conclusion that $\phi(\varpi\xi) = \phi(\varpi)\phi(\xi)$ as required. □

Theorem 3.9 leads to a consequence which characterizes NRPs in-terms of identification as shown in the next corollary.

Corollary 3.10. *Every $\chi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ is a scalar multiplication map.*

Proof. It suffices to show that there exists a scalar $\gamma \in \mathbb{C}$ such that $\chi(\varpi) = \gamma\varpi$, for all $\varpi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$. By definition we have that $w(\varpi) = \sup_{\omega \in S(\mathcal{B}(\mathcal{H})_{NRP}^{NUBA})^*)} |\omega(\varpi)|$, where $S(\mathcal{B}(\mathcal{H})_{NRP}^{NUBA})^*)$ is the unit sphere in the dual space $(\mathcal{B}(\mathcal{H})_{NRP}^{NUBA})^*$. From Theorem 3.9, $w(\varpi)$ coincides with $r(\varpi)$. That is, $w(\varpi) = r(\varpi) = \sup\{|\gamma| : \gamma \in \sigma(\varpi)\}$. But χ is a NRP, so $r(\chi(\varpi)) = r(\varpi)$, showing that χ preserves $r(\varpi)$ which is sub-multiplicative, that is $r(\varpi\xi) \leq r(\varpi)r(\xi)$. Now, for every $\varpi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$, we have that $\chi(\varpi)$ must be a scalar multiple of ϖ. Therefore, $\exists \quad \gamma \in \mathbb{C}$ such that $\chi(\varpi) = \gamma\varpi, \quad$ for all $\varpi \in \mathcal{B}(\mathcal{H})_{NRP}^{NUBA}$ as required. □

4 Conclusion

This note has considered numerical radius preserver problem which is part of the LPP which have been considered over decades. However, this problem still remains interesting as it has not been solved in totality. Only partial solutions have been obtained in special cases like unital Banach algebras and C*-algebras among others. In this paper, we have characterized certain properties of numerical radius preservers in nonunital Banach algebras. In particular, we have shown that every numerical radius preserver is surjective, continuous and real linear. Moreover, they preserve multiplicativity, commutativity, involution and is a $*$-isomorphism.

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